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SOME ELEMENTARY RESULTS IN INTUITIONISTIC MODEL THEORY

WIM VELDMAN AND FRANK WAALDIJK

Abstract. We establish constructive refinements of several well-known theorems in elementary model theory. The additive group of the real numbers may be embedded elementarily into the additive group of pairs of real numbers, constructively as well as classically.

Introduction. Intuitionistic model theory, as we understand it, is part of intuitionistic mathematics. We study intuitionistic structures from the model-theoretic point of view in an intuitionistic way. We are *not* trying to find non-intuitionistic interpretations of formally intuitionistic theories.

The paper is divided into eight sections. In Section 1 we notice that notions such as "elementary equivalence" and "elementary substructure" have a straightforward constructive meaning. We classify formulas according to their quantifier-depth and define corresponding refinements of the basic model-theoretic concepts. We introduce *strongly homogeneous* structures, that is, structures with the property that every local isomorphism extends to an automorphism of the structure. We also introduce the weaker notion of a *back-and-forth-homogeneous* structure. We prove a theorem that will help us to find elementary substructures of back-and-forth-homogeneous structures.

In Section 2 we recapitulate the intuitionistic construction of the continuum and prove that the structure $(\mathbb{R}, <)$ is strongly homogeneous.

In Section 3, we consider subsets A of $\mathbb{R}$ such that $(A, <)$ is an elementary substructure of $(\mathbb{R}, <)$. We recover and extend the most important results of [7]. In Section 4, we prove that intuitionistic Baire space $(\mathscr{N}, \#_0)$ (the universal spread), considered as a set with an apartness relation, not with an order relation, is strongly homogeneous, and we mention some applications of this result. Section 5 is our first *intuitionistic intermezzo*. We discuss some consequences of the continuity principle, and show that the apartness structure $(\mathbb{R}, \#)$ is *not* strongly homogeneous. In Section 6 we prove that $(\mathbb{R}, \#)$ is back-and-forth-homogeneous. In Section 7, our second *intuitionistic intermezzo*, we show that Fraïssé's characterization of elementary equivalence is not valid constructively. In Section 8 we study $(\mathbb{R}, +)$, the additive group of the real numbers, and consider several structures that are elementarily equivalent to $(\mathbb{R}, +)$.

Most of our proofs, although intuitionistically correct, may count as "classical" proofs of the corresponding "classical" theorems. Nobody will find fault with our avoidance of indirect arguments. In the proof of Theorem 3.3.4 we use a version of the axiom of countable choice. This version of the axiom of countable choice is

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accepted in intuitionistic mathematics, and, as one may see for instance at page 15 of [5], it is also accepted in the wider circle of constructive mathematics. Specifically intuitionistic axioms are used in Sections 5, 7 and 8.3, but nowhere else.

To a large extent our constructive treatment of model-theoretic questions was inspired by the work of R. Fraïssé who, although reasoning classically, sought for direct arguments in model theory.

"Iff" means "if and only if".

We hereby express our thanks to the referee, whose comments led to some improvements of the paper, especially in Section 1.

§1. On intuitionistic model theory.

1.0. In this paper we study mathematical structures $\mathfrak{A} = (A, R, a_0, a_1, \ldots, a_{m-1})$ where A is a set, R is a relation on A, either a 1-ary relation, or a 2-ary relation or a 3-ary relation, and $(a_0, a_1, \ldots, a_{m-1})$ is a finite sequence of elements of A, so-called *constant* elements of A. The signature (or similarity type) of such a structure will be given by a pair of natural numbers (i, m) where i denotes the arity of the relation and m the length of the sequence of constant elements. Structures $\mathfrak{A}, \mathfrak{B}$ with the same signature will be called *similar* structures.

If $\mathfrak{A} = (A, R, a_0, a_1, \ldots, a_{m-1})$ is a structure and $b_0, b_1, \ldots, b_{n-1}$ is a finite sequence of elements of $\mathfrak{A}$, we will denote the structure $(A, R, a_0, a_1, \ldots, a_{m-1}, b_0, b_1, \ldots, b_{n-1})$ by $(\mathfrak{A}, b_0, b_1, \ldots, b_{n-1})$.

To each pair (i, m) corresponds a first-order-language $L_{(i,m)}$. The mathematical symbols of $L_{(i,m)}$ are an i-ary relation symbol P and m distinct individual constants: $\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{m-1}$.

The formulas of the language $L_{(i,m)}$ are built up in the usual way, from the mathematical symbols, the logical symbols and several auxiliary symbols such as brackets and parentheses. Among the logical symbols of the language are the connectives $\wedge, \vee, \rightarrow$ and $\neg$ and the quantifiers $\forall, \exists$. Intuitionistically, it is impossible to define any of them in terms of the others. The languages $L_{(i,m)}$ do *not* contain an equality symbol.

$x_0, x_1, x_2, \ldots$ is the list of individual variables of the language $L_{(i,m)}$. Free and bound occurrences of a variable x_i in a formula ϕ are defined as usual. "A formula $\phi = \phi(x_0, x_1, \ldots, x_{n-1})$" means "a formula ϕ such that every individual variable that occurs freely in ϕ is one of the variables $x_0, x_1, \ldots, x_{n-1}$."

1.1. Let $\mathfrak{A} = (A, R, a_0, a_1, \ldots, a_{m-1})$ be a mathematical structure. We define, for every formula $\phi = \phi(x_0, x_1, \ldots, x_{n-1})$ from the appropriate first-order-language and every finite sequence $b_0, b_1, \ldots, b_{n-1}$ of elements of A, the statement:

$$\mathfrak{A} \models \phi[b_0, b_1, \ldots, b_{n-1}] \qquad (\text{"}\phi \text{ is true in the structure } \mathfrak{A} \text{ if we interpret } x_0 \text{ by } b_0, x_1 \text{ by } b_1, \ldots, x_{n-1} \text{ by } b_{n-1}\text{"})$$

just as Tarski did, with the important proviso that connectives and quantifiers are interpreted constructively. Observe that, for every formula $\phi = \phi(x_0, x_1, \ldots, x_{n-1})$ and every finite sequence $b_0, b_1, \ldots, b_{n-1}$ of elements of A:

$$\mathfrak{A} \models \phi[b_0, b_1, \ldots, b_{n-1}] \text{ iff } (\mathfrak{A}, b_0, b_1, \ldots, b_{n-1}) \models \phi'$$

where ϕ' is the sentence from the language $L_{(i,n+m)}$ that we get from ϕ by replacing, for every $i < n$, every free occurrence of the individual variable x_i by the individual constant $\mathbf{c}_{m+i}$.

The following lemma is easy.

LEMMA 1.2. *Let $\mathfrak{A} = (A, \ldots)$ and $\mathfrak{B} = (B, \ldots)$ be similar structures.*

Let $\phi = \phi(x)$ be a formula from their common first-order-language. Suppose: for every a in A there exists b in B such that: $\mathfrak{A} \models \phi[a]$ iff $\mathfrak{B} \models \phi[b]$.

Then:

(i) $$\text{If } \mathfrak{A} \models \exists x[\phi(x)], \text{ then } \mathfrak{B} \models \exists x[\phi(x)]$$

and

(ii) $$\text{If } \mathfrak{B} \models \forall x[\phi(x)], \text{ then } \mathfrak{A} \models \forall x[\phi(x)].$$

PROOF. The proof is a straightforward application of the truth definition. ⊣

1.3. Let $\mathfrak{L}$ be one of our first-order-languages. To any formula ϕ from $\mathfrak{L}$ we associate a natural number, $QD(\phi)$, the *quantifier-depth* of ϕ. If ϕ has no quantifiers, then $QD(\phi) = 0$. Further, $QD(\forall y[\phi]) = QD(\exists y[\phi]) = QD(\phi) + 1$ and $QD(\phi \wedge \psi) = QD(\phi \vee \psi) = QD(\phi \rightarrow \psi) = \mathrm{Max}(QD(\phi), QD(\psi))$. Finally, $QD(\neg\phi) = QD(\phi)$.

1.4. We define some model-theoretic notions. Let $\mathfrak{A} = (A, R, a_0, a_1, \ldots, a_{m-1})$ and $\mathfrak{B} = (B, S, b_0, b_1, \ldots, b_{m-1})$ be similar structures. For each natural number n, $\mathfrak{A}$ is *n-elementarily equivalent* to $\mathfrak{B}$ (notation: $\mathfrak{A} \equiv_n \mathfrak{B}$) iff for every sentence ϕ from their common first-order-language: if $QD(\phi) \leq n$, then $\mathfrak{A} \models \phi$ iff $\mathfrak{B} \models \phi$.

$\mathfrak{A}$ is *elementarily equivalent* to $\mathfrak{B}$ (notation: $\mathfrak{A} \equiv \mathfrak{B}$) iff for each n : $\mathfrak{A} \equiv_n \mathfrak{B}$.

For each natural number n, $\mathfrak{A}$ is an *n-elementary substructure* of $\mathfrak{B}$ (notation: $\mathfrak{A} \prec_n \mathfrak{B}$) iff $A \subseteq B$ and for each finite sequence $(c_0, c_1, \ldots, c_{p-1})$ of elements of $\mathfrak{A}$: $(\mathfrak{A}, c_0, c_1, \ldots, c_{p-1}) \equiv_n (\mathfrak{B}, c_0, c_1, \ldots, c_{p-1})$.

$\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$ iff $\mathfrak{A}$ is a 0-elementary substructure of $\mathfrak{B}$.

$\mathfrak{A}$ is an *elementary substructure* of $\mathfrak{B}$ (notation: $\mathfrak{A} \prec \mathfrak{B}$) iff for each n: $\mathfrak{A} \prec_n \mathfrak{B}$.

LEMMA 1.5 (Fraissé's Lemma). *Let $\mathfrak{A} = (A, \ldots)$ and $\mathfrak{B} = (B, \ldots)$ be similar structures. Then, for each natural number n:*

If $\forall a \in A \exists b \in B[(\mathfrak{A}, a) \equiv_n (\mathfrak{B}, b)]$ and $\forall b \in B \exists a \in A[(\mathfrak{A}, a) \equiv_n (\mathfrak{B}, b)]$, then $\mathfrak{A} \equiv_{n+1} \mathfrak{B}$.

PROOF. The lemma follows easily from Lemma 1.2. ⊣

In our second Intuitionistic Intermezzo, Section 7, we show that the converse of Lemma 1.5, classically part of a famous result of Fraïssé's, (see [1]), fails constructively.

1.6. In this subsection, we introduce *strongly homogeneous* and *back-and-forth-homogeneous* structures. We start with some preliminary definitions.

1.6.1. Let $\mathfrak{A} = (A, R, a_0, a_1, \ldots, a_{m-1})$ be a structure. Without loss of generality we assume that R is a binary relation on A. We define an equivalence relation $\sim_{\mathfrak{A}}$ on the set A, as follows: for all $a, b \in A$: $a \sim_{\mathfrak{A}} b$ iff for every c in A: cRa iff cRb, and aRc iff bRc.

$\sim_{\mathfrak{A}}$ is the finest equivalence relation on the set A that respects the relation R.

Given any subset f of $A \times B$ we define $\overline{f}$ to be the closure of f under the equivalence relations $\sim_{\mathfrak{A}}$ and $\sim_{\mathfrak{B}}$, that is, $\overline{f}$ consists of all pairs (c, d) in $A \times B$ such that there exists c', d' in A, B respectively with the property: $c \sim_{\mathfrak{A}} c'$ and $d \sim_{\mathfrak{B}} d'$ and (c', d') belongs to f.

1.6.2. Let $\mathfrak{A} = (A, R, a_0, a_1, \ldots, a_{m-1})$ and $\mathfrak{B} = (B, S, b_0, b_1, \ldots, b_{m-1})$ be similar structures. We assume, without loss of generality, that R, S are binary relations on A, B respectively. A subset f of $A \times B$ is called a *local isomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ iff for all pairs $(c_0, d_0), (c_1, d_1)$ in f : $c_0 R c_1$ iff $d_0 S d_1$ and for all $i < m$: (a_i, b_i) belongs to $\overline{f}$.

We use this definition of "local isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$" as equality is not mentioned in the language of our structures. There are many situations in constructive mathematics where equality is a defined notion and not a primitive one.

For this reason, our notion of a local isomorphism from $\mathfrak{A}$ to itself slightly differs from Fraïssé's notion, see [1].

A local isomorphism f from $\mathfrak{A}$ to $\mathfrak{B}$ is called a *homomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ iff for each c in A there exists d in B such that (c, d) belongs to $\overline{f}$. A homomorphism is called an *isomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$, if, in addition, for each d in B there exists c in A such that (c, d) belongs to $\overline{f}$.

Observe that, if $A \subseteq B$, then $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ iff the identity mapping from A to B, seen as a set of ordered pairs, is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

Observe that, if f is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, then for all $(c_0, d_0), (c_1, d_1)$ in f: If $d_0 \sim_{\mathfrak{B}} d_1$, then $c_0 \sim_{\mathfrak{A}} c_1$.

Observe that, if f is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ then for all $(c_0, d_0), (c_1, d_1)$ in $\overline{f}$: If $c_0 \sim_{\mathfrak{A}} c_1$, then $d_0 \sim_{\mathfrak{B}} d_1$.

An isomorphism from $\mathfrak{A}$ to itself is called an *automorphism* of $\mathfrak{A}$.

1.6.3. Let $\mathfrak{A} = (A, \ldots)$ be a structure. We say that $\mathfrak{A}$ is *strongly homogeneous* iff for all finite sequences $(c_0, c_1, \ldots, c_{p-1})$ and $(d_0, d_1, \ldots, d_{p-1})$ of elements of $\mathfrak{A}$ of equal length: If $(\mathfrak{A}, c_0, c_1, \ldots, c_{p-1}) \equiv_0 (\mathfrak{A}, d_0, d_1, \ldots, d_{p-1})$, then there exists an automorphism f of $\mathfrak{A}$ such that for all $i < p$: (c_i, d_i) belongs to $\overline{f}$.

Observe that, if $(\mathfrak{A}, c_0, c_1, \ldots, c_{p-1}) \equiv_0 (\mathfrak{A}, d_0, d_1, \ldots, d_{p-1})$, then the set $\{(c_0, d_0), (c_1, d_1), \ldots, (c_{p-1}, d_{p-1})\}$ is a local isomorphism from $\mathfrak{A}$ to itself.

1.6.4. Let $\mathfrak{A} = (A, \ldots)$ be a structure. We say that $\mathfrak{A}$ is *back-and-forth-homogeneous* iff for all finite sequences $(c_0, c_1, \ldots, c_{p-1})$ and $(d_0, d_1, \ldots, d_{p-1})$ of elements of A of equal length: If $(\mathfrak{A}, c_0, c_1, \ldots, c_{p-1}) \equiv_0 (\mathfrak{A}, d_0, d_1, \ldots, d_{p-1})$, then for every c in A there exists d in A such that $(\mathfrak{A}, c_0, c_1, \ldots, c_{p-1}, c) \equiv_0 (\mathfrak{A}, d_0, d_1, \ldots, d_{p-1}, d)$.

Observe that it follows from Lemma 1.5 that, for each structure $\mathfrak{A}$, if $\mathfrak{A}$ is back-and-forth-homogeneous, then for all finite sequences $(c_0, c_1, \ldots, c_{p-1})$ and $(d_0, d_1, \ldots, d_{p-1})$ of elements of A of equal length, if $(\mathfrak{A}, c_0, c_1, \ldots, c_{p-1}) \equiv_0 (\mathfrak{A}, d_0, d_1, \ldots, d_{p-1})$ then $(\mathfrak{A}, c_0, c_1, \ldots, c_{p-1}) \equiv_1 (\mathfrak{A}, d_0, d_1, \ldots, d_{p-1})$ (and also, for each n in $\mathbb{N}$, $(\mathfrak{A}, c_0, c_1, \ldots, c_{p-1}) \equiv_n (\mathfrak{A}, d_0, d_1, \ldots, d_{p-1})$).

Observe that each strongly homogeneous structure (see Definition 1.6.3) is back-and-forth-homogeneous.

THEOREM 1.6.5. *Let $\mathfrak{A} = (A, \ldots)$ be a back-and-forth-homogeneous structure and let $\mathfrak{B} = (B, \ldots)$ be a substructure of $\mathfrak{A}$ such that for every finite sequence*

$(b_0, b_1, \ldots, b_{p-1})$ of elements of B and every element a of A there exists an element b of B such that $(\mathfrak{A}, b_0, b_1, \ldots, b_{p-1}, a) \equiv_0 (\mathfrak{A}, b_0, b_1, \ldots, b_{p-1}, b)$. Then: $\mathfrak{B} \prec \mathfrak{A}$.

PROOF. One proves by induction that for each natural number n, for every finite sequence $(b_0, b_1, \ldots, b_{p-1})$ of elements of B: $(\mathfrak{B}, b_0, b_1, \ldots, b_{p-1}) \equiv_n (\mathfrak{A}, b_0, b_1, \ldots, b_{p-1})$.

The induction step uses Lemma 1.5 and the remark in Section 1.6.4. ⊣

§2. The structure $(\mathbb{R}, <)$ as an example of a strongly homogeneous structure.

2.0. We define a *real number* as a pair $\alpha = (\alpha', \alpha'')$ of functions from the set $\mathbb{N}$ of natural numbers to the set $\mathbb{Q}$ of rational numbers such that

$$\forall n \in \mathbb{N}[\alpha'(n) \leq \alpha'(n+1) \leq \alpha''(n+1) \leq \alpha''(n)]$$

and

$$\forall q \in \mathbb{Q} \forall r \in \mathbb{Q}[q < r \rightarrow \exists n \in \mathbb{N}[q < \alpha'(n) \vee \alpha''(n) < r]].$$

(The latter condition is equivalent to: $\forall m \in \mathbb{N} \exists n \in \mathbb{N}[\alpha''(n) - \alpha'(n) < \frac{1}{2^m}]$.)

The set of all real numbers will be denoted by $\mathbb{R}$.

Intuitionistically, this 'set' is introduced as a subspecies of the *spread* of all functions from $\mathbb{N}$ to $\mathbb{Q} \times \mathbb{Q}$. We will not go into details, but the reader may consult [2] or [6] or [4].

2.1. We define a binary relation $<$ on $\mathbb{R}$, as follows:

$$\text{for all } \alpha, \beta \text{ in } \mathbb{R} : \alpha < \beta \text{ iff } \exists n \in \mathbb{N}[\alpha''(n) < \beta'(n)].$$

As is well-known, this relation $<$ is *co-transitive*, that is, for all α, β, γ in $\mathbb{R}$: if $\alpha < \beta$ then either $\alpha < \gamma$ or $\gamma < \beta$. We define a function i from $\mathbb{Q}$ to $\mathbb{R}$ by: for each q in $\mathbb{Q} : i(q) = (q', q'')$ where, for each n in $\mathbb{N}$, $q'(n) = q''(n) = q$.

Observe that this function i is a homomorphism from $(\mathbb{Q}, <)$ to $(\mathbb{R}, <)$ and also, for instance, from $(\mathbb{Q}, <, 0)$ to $(\mathbb{R}, <, i(0))$. Observe that for all α, β in $\mathbb{R}$: if $\alpha < \beta$ then $\exists q \in \mathbb{Q}[\alpha < i(q)$ and $i(q) < \beta]$, that is, $\mathbb{Q}$ is embedded densely into $\mathbb{R}$.

LEMMA 2.2. *Let f be an automorphism of the structure $(\mathbb{Q}, <)$, that is, an order-preserving mapping from the set of rationals onto itself.*

There exists an automorphism f^* of the structure $(\mathbb{R}, <)$ such that for every q in $\mathbb{Q} : f^*(i(q)) = i(f(q))$, that is, $f^* \circ i = i \circ f$.

PROOF. Let f be the given automorphism of the structure $(\mathbb{Q}, <)$. We define its extension f^* to $\mathbb{R}$ as follows: for each real number $\alpha = (\alpha', \alpha'') : f^*(\alpha) := (f \circ \alpha', f \circ \alpha'')$. One verifies easily that f^* has the required properties. ⊣

2.3. We define binary relations $\leqq$, # and $\equiv$ on $\mathbb{R}$, as follows: for all α, β in $\mathbb{R}$:

$\alpha \leqq \beta$ iff $\neg(\beta < \alpha)$, that is $\forall n \in \mathbb{N}[\alpha'(n) \leqq \beta''(n)]$

$\alpha \# \beta$ ("α lies apart from β") iff $\alpha < \beta$ or $\beta < \alpha$

$\alpha \equiv \beta$ ("α coincides with β, α is equal to β") iff $\neg(\alpha \# \beta)$, that is, $\alpha \leqq \beta$ and $\beta \leqq \alpha$.

Observe that $\equiv$ is the equivalence relation belonging to the structure $(\mathbb{R}, <)$ in the sense of Section 1.6.1: for all α, β in $\mathbb{R}$:

$$\alpha \equiv \beta \text{ iff for every } \gamma \text{ in } \mathbb{R}: \gamma < \alpha \text{ iff } \gamma < \beta \text{ and } \alpha < \gamma \text{ iff } \beta < \gamma.$$

Observe also that the relation # is co-transitive:

$$\text{for all } \alpha, \beta, \gamma \text{ in } \mathbb{R}\text{: if } \alpha\#\beta, \text{ then either } \alpha\#\gamma \text{ or } \gamma\#\beta.$$

We formulate an extension of Lemma 2.2.

LEMMA 2.4. *Let f be a local isomorphism from the structure $(\mathbb{Q}, <)$ to itself such that both the domain of f, $\mathrm{Dom}(f)$, and the range of f, $\mathrm{Ran}(f)$ are dense in $\mathbb{Q}$.*

There exists an automorphism f^* of the structure $\langle \mathbb{R}, < \rangle$ such that $\forall q \in \mathrm{Dom}(f)$ $[f^*(i(q)) \equiv i(f(q))]$.

(A subset A of $\mathbb{Q}$ is called dense in $\mathbb{Q}$ iff $\forall q, r \in \mathbb{Q}$ [If $q < r$, then $\exists a \in A[q < a < r]$].)

PROOF. The proof is similar to the proof of Lemma 2.2. Observe that, for every subset A of $\mathbb{Q}$ that is dense in $\mathbb{Q}$, every real number coincides with a real number $\beta = (\beta', \beta'')$ such that both β' and β'' are functions from $\mathbb{N}$ to A. ⊣

LEMMA 2.5. *Let $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ be a finite sequence of real numbers. Define a subset A of $\mathbb{Q}$ as follows: $A := \{q \in \mathbb{Q} \mid \forall j < m[i(q) < \alpha_j \vee \alpha_j < i(q)]\}$. Then:*

(i) *There exists a function $\gamma : \mathbb{N} \to \mathbb{Q}$ such that $A = \{\gamma(n) \mid n \in \mathbb{N}\}$, that is: A is an enumerable subset of $\mathbb{Q}$.*

(ii) *Every finite sequence $(b_0, b_1, \ldots, b_m)$ of $m+1$ different rational numbers contains at least one member of A.*

(iii) *A is dense in $\mathbb{Q}$.*

PROOF.

(i) Let $q_0, q_1, \ldots$ be an enumeration of the set $\mathbb{Q}$ of rational numbers. Let p be an element of A.

For each n in $\mathbb{N}$, let n_0 and n_1 be the natural numbers such that $n = 2^{n_0}(2n_1+1)-1$.

We define the function γ as follows. For each n in $\mathbb{N}$ if $\forall j < m[q_{n_0} < \alpha'_j(n_1) \vee \alpha''_j(n_1) < q_{n_0}]$, then $\gamma(n) := q_{n_0}$, if not, then $\gamma(n) := p$.

It is easy to see that γ enumerates A.

(ii) One may prove by induction that for each natural number $m > 0$, if $(\beta_0, \beta_1, \ldots, \beta_m)$ and $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ are finite sequences of real numbers of length $m+1$ and m, respectively, and $\bigwedge_{i<j\leq m} \beta_i \# \beta_j$, then $\bigvee_{i\leq m} \bigwedge_{j<m} \beta_i \# \alpha_j$.

If $m = 1$, use the co-transitivity of the apartness relation. In the induction step, use a simple combinatorial argument.

(iii) Follows from (ii). ⊣

THEOREM 2.6 (The structure $(\mathbb{R}, <)$ is strongly homogeneous). *Let $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ and $(\beta_0, \beta_1, \ldots, \beta_{m-1})$ be finite sequences of real numbers such that $(\mathbb{R}, <, \alpha_0, \alpha_1, \ldots, \alpha_{m-1}) \equiv_0 (\mathbb{R}, <, \beta_0, \beta_1, \ldots, \beta_{m-1})$ that is: $\forall j < m \forall k < m[\alpha_j < \alpha_k$ iff $\beta_j < \beta_k]$.*

There exists an automorphism f of the structure $(\mathbb{R}, <)$ such that $\forall j < m[f(\alpha_j) \equiv \beta_j]$.

PROOF. Consider $A := \{q \in \mathbb{Q} \mid \forall j < m[i(q) < \alpha_j \vee \alpha_j < i(q)\}$ and $B := \{q \in \mathbb{Q} \mid \forall j < m[i(q) < \beta_j \vee \beta_j < i(q)\}$.

If p belongs to A and r belongs to B we say that p is *similar* to r iff $\forall j < m[i(p) < \alpha_j$ iff $i(r) < \beta_j]$.

Observe that, given any p in A, there exists at least one r in B that is similar to p, and therefore infinitely many, as $\forall j < m \forall k < m[\alpha_j < \alpha_k$ iff $\beta_j < \beta_k]$. Likewise, given any r in B, there exist infinitely many p in A that are similar to r.

We also know, from Lemma 2.5, that both A and B are enumerable subsets of $\mathbb{Q}$. It is possible, therefore, to establish, by a Cantor back-and-forth-argument, a mapping g from A onto B that is order-preserving and such that for every p in A, p is similar to $g(p)$. According to Lemma 2.4 g extends to an automorphism f of the structure $(\mathbb{R}, <)$. It is easy to see that $\forall j < m[f(\alpha_j) \equiv \beta_j]$ as $\forall j < m \forall q \in A[i(q) < \alpha_j$ iff $i(g(q)) < \beta_j]$. ⊣

§3. Some elementary substructures of $(\mathbb{R}, <)$.

3.0. In this section we consider classes of subsets A of $\mathbb{R}$ with the property $(A, <) \prec (\mathbb{R}, <)$.

3.1. We call a subset A of $\mathbb{R}$ an *open* subset of $\mathbb{R}$ iff $\forall \alpha \in A \exists \beta \in \mathbb{R} \exists \gamma \in \mathbb{R}[\beta < \alpha < \gamma \wedge \forall \delta \in \mathbb{R}[\beta < \delta < \gamma \rightarrow \delta \in A]]$. Observe that, for each open subset A of $\mathbb{R}$, for each α in A, for each ζ in $\mathbb{R}$, either $\alpha \# \zeta$ or $\zeta \in A$. (Determine β, γ in $\mathbb{R}$ such that $\beta < \alpha < \gamma$ and $\forall \delta \in \mathbb{R}[\beta < \delta < \gamma \rightarrow \delta \in A]$. Then: $\beta < \zeta \vee \zeta < \alpha$ and $\alpha < \zeta \vee \zeta < \gamma$. Therefore, either: $\zeta < \alpha$ or $\alpha < \zeta$ or: $\beta < \zeta < \gamma$, that is: either $\alpha \# \zeta$ or $\zeta \in A$.)

Similarly, for each open subset A of $\mathbb{R}$, for each finite sequence $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ of elements of A, for each ζ in $\mathbb{R}$, either $\forall j < m[\alpha_j \# \zeta]$ or $\zeta \in A$.

We call a subset A of $\mathbb{R}$ *inhabited* iff $\exists \alpha \in \mathbb{R}[\alpha \in A]$, that is, iff we are able to indicate at least one element of A.

THEOREM 3.1.1. *Let A be an open and inhabited subset of $\mathbb{R}$. Then $(A, <) \prec (\mathbb{R}, <)$.*

PROOF. According to Theorems 1.6.5 and 2.6 it suffices to show: for each finite sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ of elements of A, each β in $\mathbb{R}$, there exists α in A such that

$$(\mathbb{R}, <, \alpha_0, \alpha_1, \ldots, \alpha_{m-1}, \alpha) \equiv_0 (\mathbb{R}, <, \alpha_0, \alpha_1, \ldots, \alpha_{m-1}, \beta).$$

Let $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ be a finite sequence of elements of A, and let β be an element of $\mathbb{R}$.

As we saw in Section 3.1 we may distinguish two cases:

Case (1). $\forall j < m[\alpha_j \# \beta]$. It is possible to find α in A such that $\forall j < m[(\alpha_j < \alpha \rightleftarrows \alpha_j < \beta) \wedge (\alpha < \alpha_j \rightleftarrows \beta < \alpha_j)]$.

Case (2). β belongs to A. We may define $\alpha := \beta$.

If $m = 0$, we need the assumption that A is inhabited. ⊣

3.1.2. For each α, β in $\mathbb{R}$ we define: $\langle \alpha, \beta \rangle := \{\gamma \in \mathbb{R} \mid \alpha < \gamma < \beta\}$, that is: $\langle \alpha, \beta \rangle$ is the open interval determined by α, β. It follows from Theorem 3.1.1 that both the set $\langle 0, 2 \rangle$ and the set $\langle 0, 1 \rangle \cup \langle 1, 2 \rangle$ give rise to an elementary substructure of $(\mathbb{R}, <)$. Therefore also $(\langle 0, 1 \rangle \cup \langle 1, 2 \rangle, <) \prec (\langle 0, 2 \rangle, <)$. This re-establishes a result contained in [8] but actually proved for the first time by Tonny Hurkens.

Theorem 3.1.1 seems to be a proper generalization of this result. It implies also, for instance, that the set $\langle 0, 2 \rangle \cup \{x \in \mathbb{R} \mid 1 < x < 3$ and "Riemann's hypothesis"$\}$ gives rise to an elementary substructure of $(\mathbb{R}, <)$.

The proof that we gave of Theorem 3.1.1 is essentially the same as the proof of its special case given in [8].

3.2. We call a subset A of $\mathbb{R}$ *dense in* $\mathbb{R}$ iff $\forall\alpha \in \mathbb{R}\forall\beta \in \mathbb{R}[\alpha < \beta \rightarrow \exists\gamma \in A[\alpha < \gamma < \beta]]$.

THEOREM 3.2.1. *Let* $A_0, A_1, \ldots$ *be a sequence of dense and open subsets of* $\mathbb{R}$. *Then:*

$$(\bigcap_{n\in\mathbb{N}} A_n, <) \prec (\mathbb{R}, <)$$

PROOF. According to Theorems 1.6.5 and 2.6 it suffices to show:

For each finite sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ of elements of $\bigcap_{n\in\mathbb{N}} A_n$, each β in $\mathbb{R}$, there exists α in $\bigcap_{n\in\mathbb{N}} A_n$ such that $(\mathbb{R}, <, \alpha_0, \alpha_1, \ldots, \alpha_{m-1}, \alpha) \equiv_0 (\mathbb{R}, <, \alpha_0, \alpha_1, \ldots, \alpha_{m-1}, \beta)$.

Let $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ be a finite sequence of elements of $\bigcap_{n\in\mathbb{N}} A_n$ and let β be an element of $\mathbb{R}$.

As we saw in the remarks preceding Theorem 3.1.1, we may decide, for each natural number n, either $\forall j < m[\alpha_j \# \beta]$ or $\beta \in A_n$.

Using an axiom of countable choice we determine a function γ from $\mathbb{N}$ to $\{0, 1\}$ such that $\forall n[\gamma(n) \leq \gamma(n+1)]$ and $\forall n[(\gamma(n) = 0 \rightarrow \beta \in A_n) \wedge (\gamma(n) = 1 \rightarrow \forall j < m[\alpha_j \# \beta])]$.

Let $(q_0, r_0), (q_1, r_1), \ldots$ be an enumeration of all pairs (q, r) of rational numbers such that $q < r$.

We now define a real number $\alpha = (\alpha', \alpha'')$, step-by-step, first defining $\alpha'(0), \alpha''(0)$, then $\alpha'(1), \alpha''(1)$, and so on. We take care that, for each natural number n:

(i) $\alpha'(n) \leq \alpha'(n+1) \leq \alpha''(n+1) \leq \alpha''(n)$.
(ii) Either $q_n < \alpha'(n)$ or $\alpha''(n) < r_n$.
(iii) $\langle i(\alpha'(n)), i(\alpha''(n))\rangle \subseteq A_n$.
(iv) If $\gamma(n) = 0$, then there exists p in $\mathbb{N}$ such that $\alpha'(n) = \beta'(p)$ and $\alpha''(n) = \beta''(p)$.
(v) If $\gamma(n) = 1$ and $\forall i < n[\gamma(i) = 0]$, then for each $j < m$, $i(\alpha'(n)) < \alpha_j$ iff $i(\alpha''(n)) < \alpha_j$ iff $\beta < \alpha_j$ and $\alpha_j < i(\alpha'(n))$ iff $\alpha_j < i(\alpha''(n))$ iff $\alpha_j < \beta$.

We leave it to the reader to verify that it is possible to construct α', α'' in such a way that these conditions are satisfied and that, if we do so, α is a real number that belongs to $\bigcap_{n\in\mathbb{N}} A_n$ and $(\mathbb{R}, <, \alpha_0, \alpha_1, \ldots, \alpha_{m-1}, \alpha) \equiv_0 (\mathbb{R}, <, \alpha_0, \ldots, \alpha_{m-1}, \beta)$.

COROLLARY 3.2.2. *Let* $\alpha_0, \alpha_1, \ldots$ *be a sequence of real numbers and let* A *be a subset of* $\mathbb{R}$ *such that* $\forall\alpha \in \mathbb{R}[\forall n \in \mathbb{N}[\alpha \# \alpha_n] \rightarrow \alpha \in A]$.

Then: $(A, <) \prec (\mathbb{R}, <)$.

PROOF. Define, for each $n \in \mathbb{N}$, $A_n := \{\alpha \in \mathbb{R} \mid \alpha \# \alpha_n\}$ and apply Theorem 3.2.1. ⊣

3.2.3. Corollary 3.2.2 implies that the set $\{\alpha \in \mathbb{R} \mid \forall q \in \mathbb{Q}[i(q) \# \alpha]\}$ gives rise to an elementary substructure of $(\mathbb{R}, <)$, a result proved in [7].

Theorem 3.2.1 seems to be a proper generalization of this result.

There is another application of Corollary 3.2.2.

Consider the open interval $I = \langle -1, 1\rangle$ and its subintervals

$$I_0 = \langle -\tfrac{1}{2}, \tfrac{1}{2}\rangle \quad I_1 = \langle \tfrac{1}{2}, \tfrac{2}{3}\rangle \quad I_{-1} = \langle -\tfrac{2}{3}, -\tfrac{1}{2}\rangle, \ldots,$$
$$I_n = \langle \tfrac{n}{n+1}, \tfrac{n+1}{n+2}\rangle, \quad I_{-n} = \langle -\tfrac{n+1}{n+2}, -\tfrac{n}{n+1}\rangle, \ldots$$

This division of the interval $\langle -1, 1\rangle$ into subintervals translates naturally into a similar subdivision of any given interval $A = \langle \alpha, \beta\rangle$ (such that $\alpha < \beta$) into intervals $A_0, A_1, A_{-1}, \ldots$ and so on.

We now associate to any non-empty finite sequence $a = (a_0, a_1, \ldots, a_{m-1})$ of integers an open interval A_a in $\mathbb{R}$, as follows:

$A_{(0)} := (-1, 1), A_{(1)} := (1, 2), A_{(-1)} := (-2, -1), A_{(2)} = (2, 3), \ldots$, and for each non-empty finite sequence $a = (a_0, \ldots, a_{m-1})$ of integers for each integer n : $A_{(a_0,\ldots,a_{m-1},n)} := (A_{(a_0,\ldots,a_{m-1})})_n$. This system of open intervals generates a mapping j of the set $\mathbb{Z}^{\mathbb{N}}$ of all infinite sequences of integers into $\mathbb{R}$: for each sequence α of integers $j(\alpha)$ is a real number that, for each natural number $m > 0$, belongs to $A_{(\alpha(0),\alpha(1),\ldots,\alpha(m-1))}$.

The range of this function j coincides with the set of all real numbers that lie apart from every endpoint of any interval A_a, and thus, according to Corollary 3.2.2, gives rise to an elementary substructure of $(\mathbb{R}, <)$.

We define a binary relation $<^*$ on the set $\mathbb{Z}^{\mathbb{N}}$: for all α, β in $\mathbb{Z}^{\mathbb{N}}$: $\alpha <^* \beta$ if and only if $\exists n[\alpha(n) < \beta(n) \wedge \forall j < n[\alpha(j) = \beta(j)]]$.

One now sees that j is an elementary embedding of the structure $(\mathbb{Z}^{\mathbb{N}}, <^*)$ into the structure $(\mathbb{R}, <)$. Therefore, $(\mathbb{Z}^{\mathbb{N}}, <^*)$ and $(\mathbb{R}, <)$ are elementarily equivalent structures.

This answers a question left open in [8].

3.2.4. The lexicographical ordering $<^*$ may be defined on the set $\mathbb{Q}^{\mathbb{N}}$ of all sequences of rationals as well as on $\mathbb{Z}^{\mathbb{N}}$: for all α, β in $\mathbb{Q}^{\mathbb{N}}$: $\alpha <^* \beta$ if and only if $\exists n[\alpha(n) < \beta(n) \wedge \forall j < n[\alpha(j) = \beta(j)]]$. The structure $(\mathbb{Q}^{\mathbb{N}}, <^*)$ also embeds elementarily into the structure $(\mathbb{R}, <)$.

We see this as follows. It suffices to embed $(\mathbb{Q}^{\mathbb{N}}, <^*)$ into $(\langle -1, 1\rangle, <)$. We leave it to the reader to associate to each rational number q an open interval $J_q = \langle j'_q, j''_q\rangle$ in $\langle -1, 1\rangle$ in such a way that (i) for each q, r in $\mathbb{Q}$: if $q < r$, then $j''_q < j'_r$ and (ii) the set $\bigcup\{J_q \mid q \in \mathbb{Q}\}$ is dense in $\langle -1, 1\rangle$ and (iii) for each q in $\mathbb{Q}$, the length of J_q, that is, $j''_q - j'_q$, is less than 1. We may construct, to any given open interval $B = \langle \beta', \beta''\rangle$ a similar system $(B_q)_{q\in\mathbb{Q}}$ of subintervals of B, such that for each q in Q, the length of B_q is less than $\frac{1}{2}(\beta'' - \beta')$.

We now associate to any non-empty finite sequence $b = (b_0, b_1, \ldots, b_{m-1})$ of rationals an open interval B_b in $\langle -1, 1\rangle$ as follows:

for each rational number q : $B_{(q)} := J_q$, and for each non-empty finite sequence $b = (b_0, b_1, \ldots, b_{m-1})$ of rationals, for each rational number q : $B_{(b_0,b_1,\ldots,b_{m-1},q)} := (B_{(b_0,b_1,\ldots,b_{m-1})})_q$.

This system of open intervals generates in the obvious way a mapping k from the set $\mathbb{Q}^{\mathbb{N}}$ to the set $\mathbb{R}$ that satisfies, for all α, β in $\mathbb{Q}^{\mathbb{N}}$: if $\alpha <^* \beta$, then $k(\alpha) < k(\beta)$.

Observe that the range of this mapping coincides with $\bigcap_{m\in\mathbb{N}} \bigcup_{b\in\mathbb{Q}^m} B_b$, and thus with a countable intersection of dense and open subsets of $\langle -1, 1\rangle$. According to Theorem 3.2.1, the range of k gives rise to an elementary substructure of $(\langle -1, 1\rangle, <)$, and therefore, k is an elementary embedding from $(\mathbb{Q}^{\mathbb{N}}, <^*)$ into

$(\langle -1, 1\rangle, <)$ and also into $(\mathbb{R}, <)$. So $(\mathbb{Q}, <^*)$ and $(\mathbb{R}, <)$ are elementarily equivalent structures and $(\mathbb{Q}^{\mathbb{N}}, <^*)$ and $(\mathbb{Z}^{\mathbb{N}}, <^*)$ are elementarily equivalent structures. The latter fact was stated without proof in [8].

One may verify that the structures $(\mathbb{Z}^{\mathbb{N}}, <^*)$ and $(\mathbb{Q}^{\mathbb{N}}, <^*)$ are both strongly homogeneous and that $(\mathbb{Z}^{\mathbb{N}}, <^*)$ is an elementary substructure of $(\mathbb{Q}^{\mathbb{N}}, <^*)$.

3.3. We will reprove and generalize some results from [7]. We introduce a third kind of elementary substructures of $(\mathbb{R}, <)$.

3.3.0. Let A be a subset of $\mathbb{R}$. We call a real number α a *left accumulation point* of A iff $\forall \gamma \in \mathbb{R}[\gamma < \alpha \rightarrow \exists \delta \in A[\gamma < \delta < \alpha]]$, we call α a *right accumulation point* of A iff $\forall \gamma \in \mathbb{R}[\alpha < \gamma \rightarrow \exists \delta \in A[\alpha < \delta < \gamma]]$, and we call α a *two-sided accumulation point* of A if α is both a left and a right accumulation point of A. We say that A is a $<$-*coherent* subset of $\mathbb{R}$ iff every element of A is a two-sided accumulation point of A. (Constructively, this is somewhat stronger than saying that A has no (left- or right-) isolated points.)

3.3.1. It is not true constructively that every $<$-coherent subset of $\mathbb{R}$ gives rise to an elementary substructure of $(\mathbb{R}, <)$. We mention two counterexamples. A first counter-example is the set $\{i(q) \mid q \in \mathbb{Q}\}$. The formula $\forall x \forall y[xPy \vee \neg(xPy)]$ holds in the structure $(\mathbb{Q}, <)$, but by a weak version of the intuitionistic continuity principle, its negation is true in $(\mathbb{R}, <)$, as we will see in Section 5. Let us define, for each α, β in $\mathbb{R}$, $[\alpha, \beta\rangle := \{\gamma \in \mathbb{R} \mid \alpha \leqq \gamma < \beta\}$, that is the left-closed-right-open interval in $\mathbb{R}$ determined by α, β. A second counterexample is the set $\langle -1, 0\rangle \cup [0, 1\rangle$. The formula $\exists x \forall y[\neg(yPx) \vee yPx]$ holds in the structure $(\langle -1, 0\rangle \cup [0, 1\rangle, <)$ whereas, again by a weak version of the intuitionistic continuity principle, its negation holds in the structure $(\mathbb{R}, <)$. (These facts are shown in [8].)

3.3.2. We call a subset A of $\mathbb{R}$ a *stable* subset of $\mathbb{R}$ iff $\forall \alpha \in \mathbb{R}[\neg\neg(\alpha \in A) \rightarrow \alpha \in A]$.

We call a subset A of $\mathbb{R}$ a *real* subset of $\mathbb{R}$ iff $\forall \alpha \in \mathbb{R} \forall \beta \in \mathbb{R}[(\alpha \in A \wedge \alpha \equiv \beta) \rightarrow \beta \in A]$.

Observe that every open subset of $\mathbb{R}$ and also every subset of $\mathbb{R}$ that is a countable intersection of open subsets of $\mathbb{R}$ is a real subset of $\mathbb{R}$.

We will show that every inhabited, real, stable and $<$-coherent subset of $\mathbb{R}$ gives rise to an elementary substructure of $(\mathbb{R}, <)$.

We need the following lemma.

LEMMA 3.3.3. *Let A be a $<$-coherent subset of $\mathbb{R}$.*
For every finite sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ of elements of A, for every β in $\mathbb{R}$:
(i) *If $\forall j < m[\alpha_j < \beta]$, then $\exists \gamma \in A \forall j < m[\alpha_j < \gamma < \beta]$*
(ii) *If $\forall j < m[\beta < \alpha_j]$, then $\exists \gamma \in A \forall j < m[\beta < \gamma < \alpha_j]$.*

PROOF. We will prove (i), leaving (ii) to the reader.

Suppose $\forall j < m[\alpha_j < \beta]$. Determine $\delta_0, \delta_1, \ldots, \delta_{m-1}$ in A such that $\forall j < m[\alpha_j < \delta_j < \beta]$. Define $\gamma_0, \gamma_1, \ldots, \gamma_{m-1}$ in A inductively, as follows: Let $\gamma_0 := \delta_0$. Observe: $\alpha_0 < \gamma_0 < \beta$ and $\alpha_1 < \delta_1 < \beta$ and distinguish two cases: *case* (1): $\alpha_1 < \gamma_0$, then take $\gamma_1 := \gamma_0$ and observe: $\alpha_0 < \gamma_1$ and $\alpha_1 < \gamma_1$ or *case* (2): $\gamma_0 < \delta_1$, then take $\gamma_1 := \delta_1$ and observe: $\alpha_0 < \gamma_1$ and $\alpha_1 < \gamma_1$.

Continue in this way. Choose one of the statements "$\alpha_2 < \gamma_1$" or "$\gamma_1 < \delta_2$" and prove it. If you chose the first one, let $\gamma_2 := \gamma_1$, if not, let $\gamma_2 := \delta_2$. And so on.

After m steps we find $\gamma := \gamma_{m-1}$ such that $\forall j < m[\alpha_j < \gamma < \beta]$ and: $\gamma \in A$.

(Observe that we have some freedom when carrying out the above instructions, as, in general, *case* (1) does not exclude *case* (2).One may however obey the following rule (we consider the definition of γ_1): wait for the first natural number n such that either $\alpha_1''(n) < \gamma_0'(n)$ or $\gamma_0''(n) < \gamma_1'(n)$, say n_0, and follow *case* (1) if $\alpha_1''(n_0) < \gamma_0'(n_0)$ and *case* (2) if *not* $\alpha_1''(n_0) < \gamma_0'(n_0)$). ⊣

THEOREM 3.3.4. *Let A be an inhabited real, stable and $<$-coherent subset of $\mathbb{R}$. Then $(A, <) \prec (\mathbb{R}, <)$.*

PROOF. According to Theorems 1.6.5 and 2.6 it suffices to show that for each finite sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ of elements of A, each β in $\mathbb{R}$, there exists α in A such that $(\mathbb{R}, <, \alpha_0, \alpha_1, \ldots, \alpha_{m-1}, \alpha) \equiv_0 (\mathbb{R}, <, \alpha_0, \alpha_1, \ldots, \alpha_{m-1}, \beta)$.

Let $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ be a finite sequence of elements of A, and let β be a real number.

We now define a real number $\alpha = (\alpha', \alpha'')$, step-by-step, first defining $\alpha'(0), \alpha''(0)$, then $\alpha'(1), \alpha''(1)$, and so on.

Step 0. We distinguish two cases.

Case (1): $\exists j < m[\alpha_j'(0) \leq \beta''(0) \wedge \beta'(0) \leq \alpha_j''(0)]$.

We define

$$\alpha'(0) = \mathrm{Min}(\{\beta'(0)\} \cup \{\alpha_j'(0) \mid j < m \mid \alpha_j'(0) \leq \beta''(0) \wedge \beta'(0) \leq \alpha_j''(0)\})$$

and

$$\alpha''(0) = \mathrm{Max}(\{\beta''(0)\} \cup \{\alpha_j''(0) \mid j < m \mid \alpha_j'(0) \leq \beta''(0) \wedge \beta'(0) \leq \alpha_j''(0)\}).$$

Case (2): $\forall j < m[\beta''(0) < \alpha_j'(0) \vee \alpha_j''(0) < \beta'(0)]$, and therefore: $\forall j < m[\beta < \alpha_j \vee \alpha_j < \beta]$. Using Lemma 3.3.3 we determine γ in A such that $\forall j < m[(\beta < \alpha_j \rightarrow \gamma < \alpha_j) \wedge (\alpha_j < \beta \rightarrow \alpha_j < \gamma)]$. We define, for each natural number n: $\alpha'(n) := \gamma'(n)$ and $\alpha''(n) = \gamma''(n)$.

Step 1. We take this step only if we did not have *Case* (2) in Step 0. Again, we distinguish two cases.

Case (1): $\exists j < m[\alpha_j'(1) \leq \beta''(1) \wedge \beta'(1) \leq \alpha_j''(1)]$.

We define

$$\alpha'(1) = \mathrm{Min}(\{\beta'(1)\} \cup \{\alpha_j'(1) \mid j < m \mid \alpha_j'(1) \leq \beta''(1) \wedge \beta'(1) \leq \alpha_j''(1)\})$$

and

$$\alpha''(1) = \mathrm{Max}(\{\beta''(1)\} \cup \{\alpha_j''(1) \mid j < m \mid \alpha_j'(1) \leq \beta''(1) \wedge \beta'(1) \leq \alpha_j''(1)\}).$$

Case (2): $\forall j < m[\beta''(1) < \alpha_j'(1) \vee \alpha_j''(1) < \beta'(1)]$, and therefore: $\forall j < m[\beta < \alpha_j \vee \alpha_j < \beta]$.

Determine $j_0 < m$ such that $\alpha'(0) \leq \alpha_{j_0}'(0) \leq \alpha_{j_0}''(0) \leq \alpha''(0)$. Without loss of generality, we may assume: $\alpha_{j_0} < \beta$. Using Lemma 3.3.3 we determine γ in A such that: $\gamma < \beta$ and $\forall j < m[(\beta < \alpha_j \rightarrow \gamma < \alpha_j) \wedge (\alpha_j < \beta \rightarrow \alpha_j < \gamma)]$ and: $\alpha'(0) \leq \gamma'(0) \leq \gamma''(0) \leq \alpha''(0)$.

We define, for each natural number n, $\alpha'(n+1) := \gamma'(n+1)$ and $\alpha''(n+1) := \gamma''(n+1)$.

And so on.

We leave it to the reader to verify that α is indeed a real number. We show that α belongs to A. Remark that, if $\forall j < m[\alpha_j < \beta \vee \beta < \alpha_j]$, then $\alpha \in A$. If, on the other hand, $\neg\forall j < m[\alpha_j < \beta \vee \beta < \alpha_j]$, then $\beta \equiv \alpha$ and $\neg\neg\exists j < m[\beta \equiv \alpha_j]$. (As this step requires some familiarity with constructive reasoning, we spell it out. Suppose $\neg\exists j < m[\beta \equiv \alpha_j]$. Then $\forall j < m[\neg(\beta \equiv \alpha_j)]$, that is, $\forall j < m\neg\neg\exists n[\alpha_j''(n) < \beta(n) \vee \beta''(n) < \alpha_j'(n)]$. This is equivalent to $\neg\neg\forall j < m\exists n[\alpha_j''(n) < \beta(n) \vee \beta''(n) < \alpha_j'(n)]$, (double negations commute with finite conjunctions), that is $\neg\neg\forall j < m[\alpha_j < \beta \vee \beta < \alpha_j]$. Contradiction. Therefore: $\neg\neg\exists j < m[\beta \equiv \alpha_j]$.)

Therefore, if $\neg\forall j < m[\alpha_j < \beta \vee \beta < \alpha_j]$, then $\neg\neg(\alpha \in A)$, and thus, $\alpha \in A$. (We use the fact that A is a real and stable subset of $\mathbb{R}$.) As $\neg\neg(\forall j < m[\alpha_j < \beta \vee \beta < \alpha_j] \vee \neg\forall j < m[\alpha_j < \beta \vee \beta < \alpha_j])$, we conclude: $\neg\neg(\alpha \in A)$, and again: $\alpha \in A$.

Finally, we prove that $(\mathbb{R}, <, \alpha_0, \alpha_1, \ldots, \alpha_{m-1}, \alpha) \equiv_0 (\mathbb{R}_0, <, \alpha_0, \alpha_1, \ldots, \alpha_{m-1}, \beta)$, that is, $\forall j < m[(\alpha_j < \alpha \rightleftarrows \alpha_j < \beta) \wedge (\alpha < \alpha_j \rightleftarrows \beta < \alpha_j)]$. We only prove that $\forall j < m[\alpha_j < \alpha \rightleftarrows \alpha_j < \beta]$, leaving the second part to the reader.

First, suppose $j_0 < m$ and $\alpha_{j_0} < \alpha$. Determine a natural number n such that $\alpha_{j_0}''(n) < \alpha'(n)$. Observe that *either* $\alpha'(n) \leq \beta'(n)$ (and therefore $\alpha_{j_0} < \beta$), *or* the construction of α has been completed at stage n or even earlier, as we discovered that $\forall j < m[\alpha_j < \beta \vee \beta < \alpha_j]$. In the latter case we know that $\alpha_{j_0} < \alpha \rightleftarrows \alpha_{j_0} < \beta$, therefore $\alpha_{j_0} < \beta$.

Next, suppose $j_0 < m$ and $\alpha_{j_0} < \beta$. By the co-transitivity of the relation $<$ we know that $\forall j < m[\alpha_{j_0} < \alpha_j \vee \alpha_j < \beta]$.

Determine a natural number n such that $\alpha_{j_0}''(n) < \beta'(n)$ and $\forall j < m[\alpha_{j_0}''(n) < \alpha_j'(n) \vee \alpha_j''(n) < \beta'(n)]$. Observe that *either* $\alpha'(n) \geq \mathrm{Min}(\{\beta'(n)\} \cup \{\alpha_j'(n) | j < m | \beta'(n) \leq \alpha_j''(n)\})$ (and therefore, as for each $j < m$, if $\beta'(n) \leq \alpha_j''(n)$, then $\alpha_{j_0}''(n) < \alpha_j'(n)$, we know that $\alpha_{j_0}''(n) < \alpha'(n)$, that is, $\alpha_{j_0} < \alpha$, *or* the construction of α has been completed at stage n or even earlier as we discovered $\forall j < m[\alpha_j < \beta \vee \beta < \alpha_j]$. In the latter case we are sure that $\alpha_{j_0} < \alpha \rightleftarrows \alpha_{j_0} < \beta$, therefore $\alpha_{j_0} < \alpha$. ⊣

COROLLARY 3.3.5. *Let A be a dense subset of $\mathbb{R}$. Then: the set $\{\alpha \in \mathbb{R} \mid \neg\neg\exists\beta \in A[\alpha \equiv \beta]\}$ gives rise to an elementary substructure of $(\mathbb{R}, <)$.*

The proof of this corollary is immediate. In particular, as was observed in [7], both the set of the not-rational and the set of the not-not-rational real numbers give rise to an elementary substructure of $(\mathbb{R}, <)$. (On this example, see also Section 5.3.)

§4. Intuitionistic Baire space $\mathcal{N}$ is also strongly homogeneous.

4.0. In this section we consider the set $\mathcal{N}$ of all infinite sequences of natural numbers, also called the universal spread or intuitionistic Baire space. An element α of $\mathcal{N}$ is a function from the set $\mathbb{N}$ of natural numbers to itself. Given elements α, β of $\mathcal{N}$ we say: $\alpha \#_0 \beta$ (α lies apart$_0$ from β) iff $\exists n \in \mathbb{N}[\alpha(n) \neq \beta(n)]$. We will see that the structure $(\mathcal{N}, \#_0)$ is strongly homogeneous in the sense of Definition 1.6.3.

4.1. $\mathbb{N}^* = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ is the set of all finite sequences of natural numbers. $*$ is the binary operation of concatenation on $\mathbb{N}^*$, that is, for all $a = \langle a(0), a(1), \ldots, a(m-1)\rangle$ and $b = \langle b(0), b(1), \ldots, b(n-1)\rangle$ in $\mathbb{N}^*$, $a * b$ denotes the finite sequence

obtained by putting b behind a, that is: $a*b = \langle a(0), a(1), \ldots, a(m-1), b(0), b(1), \ldots, b(n-1)\rangle$.

Given a, b in $\mathbb{N}^*$, we say: $a \sqsubseteq b$ ("the finite sequence a is an initial part of the finite sequence b") iff $\exists c \in \mathbb{N}^*[a * c = b]$.

Given α in $\mathbb{N}$, n in $\mathbb{N}$, we define $\overline{\alpha}(n) := \langle \alpha(0), \alpha(1), \ldots, \alpha(n-1)\rangle$, that is the finite sequence of length n that is an initial part of the infinite sequence α.

LEMMA 4.2. *Let* $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ *and* $(\beta_0, \beta_1, \ldots, \beta_{m-1})$ *be finite sequences of elements of* $\mathcal{N}$ *such that* $\forall j < m \forall k < m[\alpha_j \#_0 \alpha_k \rightleftarrows \beta_j \#_0 \beta_k]$. *Then* $\forall n \exists p > n \forall j < m \forall k < m[\overline{\alpha}_j(p) = \overline{\alpha}_k(p) \rightleftarrows \overline{\beta}_j(p) = \overline{\beta}_k(p)]]$.

PROOF. Define, for each n in $\mathbb{N}$:

$$A_n := \{(j,k) \mid j < m, k < m \mid \overline{\alpha}_j(n) \neq \overline{\alpha}_k(n)\}$$
$$B_n := \{(j,k) \mid j < m, k < m \mid \overline{\beta}_j(n) \neq \overline{\beta}_k(n)\}$$

The assumption implies:

$$\forall n \in \mathbb{N} \exists p \in \mathbb{N}[p > n \wedge (A_n = B_n \vee A_n \neq A_p \vee B_n \neq B_p)].$$

Observe that, for each n in $\mathbb{N}$, $A_n \subseteq A_{n+1}$ and $B_n \subseteq B_{n+1}$, and A_n and B_n are decidable subsets of $\{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, m-1\}$. Therefore $\forall n \in \mathbb{N} \exists p \in \mathbb{N}[p > n \wedge A_p = B_p]$. ⊣

THEOREM 4.3 (The structure $(\mathcal{N}, \#_0)$ is strongly homogeneous). *Let* $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ *and* $(\beta_0, \beta_1, \ldots, \beta_{m-1})$ *be finite sequences of elements of* $\mathcal{N}$ *such that*

$$(\mathcal{N}, \#_0, \alpha_0, \alpha_1, \ldots, \alpha_{m-1}) \equiv_0 (\mathcal{N}, \#_0, \beta_0, \beta_1, \ldots, \beta_{m-1}),$$

that is:

$$\forall j < m \forall k < m[\alpha_j \#_0 \alpha_k \rightleftarrows \beta_j \# \beta_k].$$

There exists an automorphism f *of the structure* $(\mathcal{N}, \#_0)$ *such that* $\forall j < m[f(\alpha_j) = \beta_j]$.

PROOF. By Lemma 4.2 there exists a strictly increasing sequence $p_0, p_1, p_2, \ldots$ of natural numbers such that $\forall n \in \mathbb{N} \forall j < m \forall k < m[\overline{\alpha}_j(p_n) = \overline{\alpha}_k(p_n) \rightleftarrows \overline{\beta}_j(p_n) = \overline{\beta}_k(p_n)]$.

Determine, for each n in $\mathbb{N}$, a permutation f_n of $\mathbb{N}^{p_n}$, the set of finite sequences of natural numbers of length p_n, in such a way that

$$\forall n \in \mathbb{N} \forall j < m[f_n(\overline{\alpha}_j(p_n)) = \overline{\beta}_j(p_n)]$$

and

$$\forall n \in \mathbb{N} \forall a \in \mathbb{N}^{p_n} \forall b \in \mathbb{N}^{p_{n+1}}[a \sqsubseteq b \to f_n(a) \sqsubseteq f_n(b)].$$

Let f be the mapping from $\mathbb{N}$ to $\mathbb{N}$ that is determined by: $\forall n \in \mathbb{N}[\overline{f(\alpha)}p_n = f_n(\overline{\alpha}(p_n))]$.

It is easy to see that f fulfils the requirements. ⊣

4.4. We determine an important class of elementary substructures of $(\mathscr{N}, \#_0)$. Let A be a subset of $\mathbb{N}$. We say that A is a $\#_0$-*coherent* subset of $\mathscr{N}$ iff $\forall \alpha \in A \forall n \in \mathbb{N} \exists \beta \in A[\beta \#_0 \alpha \wedge \overline{\beta} n = \overline{\alpha} n]$.

THEOREM 4.5. *Let A be an inhabited, stable and $\#_0$-coherent subset of $\mathscr{N}$. Then $(A, \#_0) \prec (\mathscr{N}, \#_0)$.*

PROOF. The proof is similar to the proof of Theorem 3.3.4. ⊣

4.6. We mention two applications of Theorem 4.5.

4.6.1. Let A be a subset of $\mathscr{N}$. We say that A is a *spread* iff there exists a decidable subset B of the set $\mathbb{N}^*$ of finite sequences of natural numbers such that the empty sequence $\langle\rangle$ belongs to B and for every $a \in \mathbb{N}^*$, if $a \in B$, then $\exists n[a * \langle n \rangle \in B]$, and $\forall \alpha \in \mathscr{N}[\alpha \in A \rightleftarrows \forall n[\overline{\alpha} n \in B]]$. It is easy to see that every spread is an inhabited and stable subset of $\mathscr{N}$. Therefore, if A is a $\#_0$-coherent spread, then $(A, \#_0) \prec (\mathscr{N}, \#_0)$.

4.6.2. It follows from Theorem 4.5 that, if A is an inhabited $\#_0$-coherent subset of $\mathscr{N}$ and $C := \{\beta \in \mathscr{N} \mid \neg\neg(\beta \in A)\}$ then $(C, \#_0) \prec (\mathscr{N}, \#_0)$.

Take for example $A := \{\alpha \in \mathscr{N} \mid \exists n \forall m > n[\alpha(m) = 0]\}$.

Intuitionistically, the structure $(A, \#_0)$ is not elementarily equivalent to the structure $(\mathscr{N}, \#_0)$ as the formula $\forall x \forall y[xPy \vee \neg(xPy)]$ is true in $(A, \#_0)$ whereas, by a weak continuity principle, see Theorem 5.1, its negation holds in $(\mathscr{N}, \#_0)$.

On the other hand, $(\{\alpha \in \mathscr{N} \mid \neg\neg\exists n \forall m > n[\alpha(m) = 0]\}, \#_0) \prec (\mathscr{N}, \#_0)$.

We will discuss this example again in Section 5.3.

§5. First intuitionistic intermezzo: the continuity principle and some of its consequences.

5.0. The famous continuity principle may be formulated as follows:

5.0.0.

CP For every subset C of $\mathscr{N} \times \mathbb{N}$:
If $\forall \alpha \in \mathscr{N} \exists n \in \mathbb{N}[C(\alpha, n)]$,
then $\forall \alpha \in \mathscr{N} \exists n \in \mathbb{N} \exists m \in \mathbb{N} \forall \beta \in \mathscr{N}[\overline{\alpha}(m) = \overline{\beta}(m) \rightarrow C(\beta, n)]$.

(Compare [4, *27.15] and $WC - \mathbb{N}$ in [6].) We will not go into an extensive justification of this principle. Roughly, the idea is that if we are able to associate effectively a natural number to every infinite sequence α of natural numbers, we will be able to do so even for infinite sequences that are the result of a step-by-step-construction. Moreover, any infinite sequence, even if it admits of a finite description, may be the result of such a step-by-step construction.

CP is sometimes called the *weak* continuity principle.

Here is an even weaker version:

5.0.1.

CP′ For all subsets A, B of $\mathscr{N}$:
If $\forall \alpha \in \mathscr{N}[A(\alpha) \vee B(\alpha)]$,
then $\forall \alpha \in \mathscr{N} \exists m \in \mathbb{N}[\forall \beta \in \mathscr{N}[\overline{\alpha}(m) = \overline{\beta}(m) \rightarrow A(\beta)] \vee$
$\forall \beta \in \mathscr{N}[\overline{\alpha}(m) = \overline{\beta}(m) \rightarrow B(\beta)]]$

CP also admits of a generalization that is easily seen to be equivalent to it in the context of elementary intuitionistic analysis:

5.0.2.

GCP For every spread A, and every subset C of $A \times \mathbb{N}$:
If $\forall\alpha \in A \exists n \in \mathbb{N}[C(\alpha, n)]$,
then $\forall\alpha \in A \exists n \in \mathbb{N} \exists m \in \mathbb{N} \forall\beta \in A[\overline{\alpha}(m) = \overline{\beta}(m) \rightarrow C(\beta, n)]$

0 is the element α of $\mathbb{N}$ such that $\forall n \in \mathbb{N}[\alpha(n) = 0]$.

Theorems 5.1 and 5.2 mention two well-known consequences of the continuity principle.

THEOREM 5.1. $\neg\forall\alpha \in \mathcal{N}[\alpha \#_0 \mathbf{0} \vee \neg(\alpha \#_0 \mathbf{0})]$.

PROOF. Suppose $\forall\alpha \in \mathcal{N}[\alpha \#_0 \mathbf{0} \vee \neg(\alpha \#_0 \mathbf{0})]$.
Applying CP′ one finds m in $\mathbb{N}$ such that $\forall\alpha \in \mathcal{N}[\overline{\alpha}(m) = \overline{\mathbf{0}}(m) \rightarrow \alpha = \mathbf{0}]$, an obvious contradiction. ⊣

An easy extension of Theorem 5.1 shows that the formulas $\forall x \neg\forall y[xPy \vee \neg(xPy)]$ and (hence) $\neg\forall x\forall y[xPy \vee \neg(xPy)]$ are valid in the structure $(\mathcal{N}, \#_0)$, a fact we referred to in Section 4.6.2. In order to obtain a similar result on the structure $(\mathbb{R}, <)$ we introduce the following notion.

Let $(q_0, r_0), (q_1, r_1), \ldots$ be a fixed enumeration of all pairs (q, r) of rational numbers such that $q < r$.

A *canonical real number* is a pair $\alpha = (\alpha', \alpha'')$ of functions from the set $\mathbb{N}$ of natural numbers to the set $\mathbb{Q}$ of rational numbers such that $\forall n \in \mathbb{N}[\alpha'(n) \leq \alpha'(n+1) < \alpha''(n+1) \leq \alpha''(n)]$ and $\forall n \in \mathbb{N}[q_n < \alpha'(n) \vee \alpha''(n) < r_n]$.

Via some coding of pairs of rational numbers by natural numbers the set $\mathbb{R}_{\text{can}}$ of the canonical real numbers may be conceived as a spread.

Let 0^* be some canonical real number α such that $\forall n \in \mathbb{N}[\alpha'(n) < 0 < \alpha''(n)]$.

THEOREM 5.2. $\neg\forall\alpha \in \mathbb{R}[\alpha < 0^* \vee \neg(\alpha < 0^*)]$.

PROOF. Suppose $\forall\alpha \in \mathbb{R}[\alpha < 0^* \vee \neg(\alpha < 0^*)]$.
Then $\forall\alpha \in \mathbb{R}_{\text{can}}[\alpha < 0^* \vee \neg(\alpha < 0^*)]$.
As $\mathbb{R}_{\text{can}}$ is a spread, we apply GCP and find m in $\mathbb{N}$ such that

$$\forall\alpha \in \mathbb{R}_{\text{can}}[(\overline{\alpha'}(m) = \overline{0^{*\prime}}(m) \wedge \overline{\alpha''}(m) = \overline{0^{*\prime\prime}}(m)) \rightarrow \neg(\alpha < 0^*)].$$

This is contradictory as there exists a canonical real number α such that $i(\frac{1}{2}0^{*\prime}(m)) \equiv \alpha$ and $\overline{\alpha'}(m) = \overline{0^{*\prime}}(m) \wedge \overline{\alpha''}(m) = \overline{0^{*\prime\prime}}(m)$. (Observe that $\frac{1}{2}0^{*\prime}(m) < 0$.) ⊣

It follows from Theorem 5.2 that the formulas

$$\exists y \neg\forall x[xPy \vee \neg(xPy)]$$

and

$$\neg\forall x\forall y[xPy \vee \neg(xPy)]$$

are true in the structure $(\mathbb{R}, <)$ whereas the formula

$$\forall x\forall y[xPy \vee \neg(xPy)]$$

is true in the structure $(\mathbb{Q}, <)$.

Observe that the formula $\forall x \neg\forall y[\neg(yPx) \vee yPx]$ is also true in $(\mathbb{R}, <)$ and that for every α in $\langle -1, 0\rangle \cup [0, 1)$ either $\alpha < 0$ or $0 \leq \alpha$. Therefore, the formula

$\exists x \forall y[\neg(yPx) \vee yPx]$ is true in $(\langle -1, 0\rangle \cup [0, 1\rangle, <)$. One might say that the latter structure contains a left-decidable point. Using left- and right-decidable points one may show that there exist, in an intuitionistically precise sense, uncountably many dense subsets of $\mathbb{R}$ that give rise to mutually elementarily different substructures of $(\mathbb{R}, <)$, see [8], where a similar result is obtained for the structure $(\mathscr{N}, <^*)$.

In a forthcoming paper by the second author a stronger result will be established: there exist uncountably many subspreads of $\mathscr{N}$ that give rise to mutually elementarily different substructures of the structure $(\mathscr{N}, \#_0)$. Observe that $\#_0$ may be defined in terms of $<^*$, but not conversely.

5.3. Consider $B := \{\alpha \in \mathscr{N} \mid \neg\neg\exists n \forall m > n[\alpha(m) = 0]\}$.

We have seen in Section 4.6.2 that $(B, \#_0) \prec (\mathscr{N}, \#_0)$.

This implies, for instance: $\neg\forall\alpha \in B \forall\beta \in B[\alpha\#_0\beta \vee \neg(\alpha\#_0\beta)]$. This is not surprising and may be proved more directly as follows. Consider $T := \{\alpha \in \mathscr{N} \mid \forall m \forall n[(\alpha(m) \neq 0 \wedge \alpha(n) \neq 0) \rightarrow m = n]\}$. One verifies easily that T is a spread, that $\mathbf{0} \in T \subseteq B$ and that $\neg\forall\alpha \in T \forall\beta \in T[\alpha\#_0\beta \vee \alpha = \beta]$, as $\neg\forall\beta \in T[\mathbf{0}\#_0\beta \vee \mathbf{0} = \beta]$. The latter fact, like Theorem 5.1, is an easy consequence of the continuity principle.

Consider $D := \{\alpha \in \mathbb{R} \mid \neg\neg\exists q \in \mathbb{Q}[\alpha \equiv i(q)]\}$. Just after Corollary 3.3.5 we observed that D gives rise to an elementary substructure of the structure $(\mathbb{R}, <)$. Therefore, $\neg\forall\alpha \in D \forall\beta \in D[\alpha < \beta \vee \neg(\alpha < \beta)]$. Again, this is not surprising. Consider $U := \{\alpha \in \mathbb{R}_{\text{can}} \mid \forall n[\alpha'(n) \leq 0 \leq \alpha''(n) \vee \exists m > 0[\alpha'(n) = \frac{1}{m} = \alpha''(n)]]\}$. U like $\mathbb{R}_{\text{can}}$ itself may be seen as a spread, and $\forall\alpha \in U[\neg\neg(\alpha \equiv 0^* \vee \exists m > 0[\alpha \equiv i(\frac{1}{m})])]$. Therefore $U \subseteq D$. Also $0^* \in U$ and $\neg\forall\alpha \in U[\alpha < 0^* \vee \neg(\alpha < 0^*)]$. The latter fact, like Theorem 5.2, is an easy consequence of the continuity principle.

5.4. Unlike $(\mathbb{R}, <)$ and $(\mathscr{N}, \#_0)$, see Theorems 2.6 and 4.3, the structure $(\mathbb{R}, \#)$ is *not* strongly homogeneous. Although $(\mathbb{R}, \#, 0, 1, 2) \equiv_0 (\mathbb{R}, \#, 0, 2, 1)$, there is no automorphism f of the structure $(\mathbb{R}, \#)$ such that $f(0) = 0, f(1) = 2$ and $f(2) = 1$. (Somewhat inaccurately, we are using 0,1,2 to denote $i(0), i(1)$ and $i(2)$, respectively.)

For suppose there is, and let f be such an automorphism. Observe that f is a strongly injective function from $\mathbb{R}$ to $\mathbb{R}$, that is $\forall\alpha \in \mathbb{R} \forall\beta \in \mathbb{R}[\alpha\#\beta \rightarrow f(\alpha)\#f(\beta)]$. In particular, $\forall\alpha \in [0, 1][f(\alpha)\#f(2)]$, therefore $\forall\alpha \in [0, 1][f(\alpha) < 1 \vee f(\alpha) > 1]$. Applying the method of successive bisection we find a sequence $(\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots$ of pairs of real numbers such that $\alpha_0 = 0, \beta_0 = 1$, for each n in $\mathbb{N}$ *either* $\alpha_{n+1} = \alpha_n$ and $\beta_{n+1} = \frac{1}{2}(\alpha_n + \beta_n)$ *or* $\beta_{n+1} = \beta_n$ and $\alpha_{n+1} = \frac{1}{2}(\alpha_n + \beta_n)$, and $f(\alpha_n) < 1, f(\beta_n) > 1$.

Let α be a real number such that $\forall n[\alpha_n < \alpha < \beta_n]$. By a famous consequence of the weak continuity principle CP, f is continuous in α and therefore $f(\alpha) \equiv 1$, but also $\alpha\#2$ and $f(2) \equiv 1$. Contradiction.

§6. The structure $(\mathbb{R}, \#)$ as an example of a back-and-forth-homogeneous structure.

6.0. We show that the structure $(\mathbb{R}, \#)$ is back-and-forth-homogeneous.

6.0.0. Let $\alpha = (\alpha', \alpha'')$ be a real number as defined in Section 2.0. For each n in $\mathbb{N}$ we define $\alpha(n) := (\alpha'(n), \alpha''(n))$. In this way, the real number α is seen as a function from $\mathbb{N}$ to $\mathbb{Q} \times \mathbb{Q}$.

We introduce some more notations.

If (a, b) and (c, d) belong to $\mathbb{Q} \times \mathbb{Q}$ and $a \leq b$ and $c \leq d$, we say

$(a, b) < (c, d)$ iff $b < c$ ("(a, b) lies to the left of (c, d)")
$(a, b) \leq (c, d)$ iff $a \leq d$ ("(a, b) does not lie to the right of (c, d)")
$(a, b) \not\approx (c, d)$ iff $b < c \vee d < a$"(a, b) lies apart from (c, d)")
$(a, b) \approx (c, d)$ iff $c \leq b \wedge a \leq d$ ("(a, b) touches (c, d)").

If E is a subset of $\{(q_0, q_1) \mid q_0 \in \mathbb{Q}, q_1 \in \mathbb{Q} \mid q_0 \leq q_1\}$ we say that $(E, \not\approx)$ is *co-transitive* iff for all a, b, c in E, if $a \not\approx b$, then either $c \not\approx a$ or $c \not\approx b$. (See the use of this term in Section 2.1.) This is equivalent to saying that the structure $(E, \approx)$ is transitive, that is, for all a, b, c in E, if $a \approx b$ and $b \approx c$, then $a \approx c$.

LEMMA 6.0.1. *Let* $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ *be a finite sequence of real numbers. Then* $\forall n \exists p > n[(\{\alpha_0(p), \alpha_1(p), \ldots, \alpha_{m-1}(p)\}, \approx)$ *is transitive*].

PROOF. This follows from the co-transitivity of the structure $(\mathbb{R}, \#)$.
Define, for each n in $\mathbb{N}$, $A_n := \{(i, j) \mid i < m, j < m \mid \alpha_i(m) \not\approx \alpha_j(m)\}$ and $C_n := \{\alpha_i(n) \mid i < m\}$, and observe $\forall n \exists p \geq n[(C_n, \not\approx)$ is co-transitive $\vee A_n \subsetneqq A_p]$. As, for each n in $\mathbb{N}$, A_n is a decidable subset of $\{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, m-1\}$, this implies $\forall n \exists p > n[(C_p, \not\approx)$ is co-transitive)]. ⊣

Another useful observation is the following analogue of Lemma 4.2.

LEMMA 6.0.2. *Let* $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ *and* $(\beta_0, \beta_1, \ldots, \beta_{m-1})$ *be finite sequences of real numbers such that* $\forall j < m \forall k < m[\alpha_j \# \alpha_k \rightleftarrows \beta_j \# \beta_k]$. *Then* $\forall n \exists p > n \forall j < m \forall k < m[\alpha_j(p) \approx \alpha_k(p) \rightleftarrows \beta_j(p) \approx \beta_k(p)]$.

PROOF. Define, for each n in $\mathbb{N}$, $A_n := \{(i, j) \mid i < m, j < m \mid \alpha_i(m) \not\approx \alpha_j(m)\}$ and $B_n := \{(i, j) \mid i < m, j < m \mid \beta_i(m) \not\approx \beta_j(m)\}$. Observe that $\forall n \exists p \geq n[A_n = B_n \vee A_n \subsetneqq A_p \vee B_n \subsetneqq B_p]$. As, for each n in $\mathbb{N}$, A_n and B_n are decidable subsets of $\{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, m-1\}$, this implies $\forall n \exists p > n[A_p = B_p]$. ⊣

LEMMA 6.0.3. *Let* $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ *and* $(\beta_0, \beta_1, \ldots, \beta_{m-1})$ *be finite sequences of real numbers such that* $\forall j < m \forall k < m[\alpha_j \# \alpha_k \rightleftarrows \beta_j \# \beta_k]$. *Then, for every* α *in* $\mathbb{R}$: $\forall n \exists p > n[(\{\alpha(p), \alpha_0(p), \alpha_1(p), \ldots, \alpha_{m-1}(p), \beta_0(p), \beta_1(p), \ldots, \beta_{m-1}(p)\}, \approx)$ *is transitive and* $\forall j < m \forall k < m[\alpha_j(p) \approx \alpha_k(p) \rightleftarrows \beta_j(p) \approx \beta_k(p)]$.

PROOF. Define, for each n in $\mathbb{N}$: $C_n := \{\alpha(n), \alpha_0(n), \alpha_1(n), \ldots, \alpha_{m-1}(n), \beta_0(n), \beta_1(n), \ldots, \beta_{m-1}(n)\}$. Using Lemma 6.2.1, we find a strictly increasing sequence $p_0, p_1, \ldots$ of natural numbers such that $\forall n[(C_{p_n}, \approx)$ is transitive]. Now define, for each n in $\mathbb{N}$: $A_n := \{(i, j) \mid i < m, j < m \mid \alpha_i(p_n) \not\approx \alpha_j(p_n)\}$ and $B_n := \{(i, j) \mid i < m, j < m \mid \beta_i(p_n) \not\approx \beta_j(p_n)\}$. Using Lemma 6.0.2 we conclude: $\forall n \exists k > n[A_k = B_k]$. ⊣

THEOREM 6.0.4 (The structure $(\mathbb{R}, \#)$ is back-and-forth-homogeneous). *Let* $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ *and* $(\beta_0, \beta_1, \ldots, \beta_{m-1})$ *be finite sequences of real numbers such that* $\forall j < m \forall k < m[\alpha_j \# \alpha_k \rightleftarrows \beta_j \# \beta_k]$. *Then for every* α *in* $\mathbb{R}$ *there exists* β *in* $\mathbb{R}$ *such that* $\forall j < m[\alpha_j \# \alpha \rightleftarrows \beta_j \# \beta]$.

PROOF. Let $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ and $(\beta_0, \beta_1, \ldots, \beta_{m-1})$ fulfil the requirements of the theorem.

Let α be a real number.

Define, for each n in $\mathbb{N}$, $C_n := \{\alpha_j(n) \mid j < m\} \cup \{\alpha(n)\} \cup \{\beta_j(n) \mid j < m\}$. Using Lemma 6.2.3 we determine a strictly increasing sequence $p_0, p_1, \ldots$ of natural numbers such that for each n in $\mathbb{N}$, (i) $(C_{p_n}, \not\approx)$ is cotransitive, and (ii) $\forall j < m \forall k < m[\alpha_j(p_n) \not\approx \alpha_k(p_n) \rightleftarrows \beta_j(p_n) \not\approx \beta_k(p_n)]$. We show how to find β in $\mathbb{R}$ such that $\forall j < m[\alpha_j \# \alpha \rightleftarrows \beta_j \# \beta]$. We determine $\beta(0), \beta(1), \ldots$ step-by-step.

Let n be a natural number and assume that $\beta(0), \beta(1), \ldots, \beta(n-1)$ have been defined.

We now define $\beta(n)$ and distinguish three cases:

Case (0): $\exists j < m[\alpha_i(p_n) \approx \alpha(p_n)]$. We define: $\beta'(n) := \mathrm{Min}\{\beta'_j(p_n) \mid j < m \mid \alpha_j(p_n) \approx \alpha(p_n)\}$ and $\beta''(n) := \mathrm{Max}\{\beta''_j(p_n) \mid j < n \mid \alpha_j(p_n) \approx \alpha(p_n)\}$.

Case (1): $\forall j < m[\alpha_j(p_n) \not\approx \alpha(p_n)]$ and either $n = 0$, or $n > 0$ and $\exists j < m[\alpha_j(p_{n-1}) \approx \alpha(p_{n-1})]$. We choose q in $\mathbb{Q}$ such that $\beta'(n-1) \leq q \leq \beta''(n-1)$ and $\forall j < m[i(q) \# \beta_j]$ and we define $\beta'(n) = \beta''(n) = q$.

Case (2): $n > 0$ and $\forall j < m[\alpha_j(p_{n-1}) \not\approx \alpha(p_{n-1})]$. We define $\beta'(n) := \beta''(n) := \beta'(n-1)$.

We have to show that β is a well-defined real number. Obviously, $\forall n[\beta'(n) \leq \beta'(n+1) \leq \beta''(n+1) \leq \beta''(n)]$. Let q, r be rational numbers such that $q < r$. Determine t in $\mathbb{Q}$ such that $q < t < r$. Determine n in $\mathbb{N}$ such that $\forall j < m[q < \beta'_j(p_n) \vee \beta''_j(p_n) < t]$ and $\forall j < m[t < \beta'_j(p_n) \vee \beta''_j(p_n) < r]$.

Then $\forall j < m \forall k < m[(r \leq \beta''_j(p_n) \wedge \beta'_k(p_n) \leq q) \rightarrow \beta_j(p_n) \not\approx \beta_k(p_n)]$. Observe that $\forall j < m \forall k < m[(\alpha(p_n) \approx \alpha_j(p_n) \wedge \alpha(p_n) \approx \alpha_k(p_n)) \rightarrow (\alpha_j(p_n) \approx \alpha_k(p_n) \wedge \beta_j(p_n) \approx \beta_k(p_n))]$.

Therefore, if $\exists j < m[\alpha_j(p_n) \approx \alpha(p_n) \wedge r \leq \beta''_j(p_n)]$, then $\forall j < m[\alpha_j(p_n) \approx \alpha(p_n) \rightarrow q < \beta'_j(p_n)]$, that is, if $\beta(n)$ was defined by *Case* (0) and $r \leq \beta''(n)$, then $q < \beta'(n)$, or, equivalently, either $\beta''(n) < r$ or $q < \beta'(n)$. But if $\beta(n)$ was defined by *Case* (1) or *Case* (2), then $\beta'(n) = \beta''(n)$, and also: either $\beta''(n) < r$ or $q < \beta'(n)$.

Therefore, β is a well-defined real number.

Now suppose $j < m$ and $\alpha_j \# \alpha$. Determine n in $\mathbb{N}$ such that $\alpha_j(p_n) \not\approx \alpha(p_n)$. We claim that $\beta_j(p_n) \not\approx \beta(n)$.

(For suppose $\beta_j(p_n) \approx \beta(n)$. Then there exists $k < m$ such that $\beta'(n) \leq \beta'_k(p_n) \leq \beta''_k(p_n) \leq \beta'' n$ and $\beta_j(p_n) \approx \beta_k(p_n)$ and $\alpha_k(p_n) \approx \alpha(p_n)$. Therefore: $\alpha_j(p_n) \approx \alpha_k(p_n)$ and $\alpha_j(p_n) \approx \alpha(p_n)$. Contradiction.)

Therefore: $\forall j < m[\alpha_j \# \alpha \rightarrow \beta_j \# \beta]$.

Finally suppose $j < m$ and $\beta_j \# \beta$. Determine n in $\mathbb{N}$ such that $\beta(n) \not\approx \beta_j(p_n)$. Looking at the definition of $\beta(n)$, we see: $\alpha(p_m) \not\approx \alpha_j(p_m)$, therefore: $\alpha_j \# \alpha$.

This shows: $\forall j < m[\beta_j \# \beta \rightarrow \alpha_j \# \alpha]$. ⊣

6.1. Observe that every subset A of $\mathbb{R}$ that gives rise to an elementary substructure of $(\mathbb{R}, <)$, also generates an elementary substructure of $(\mathbb{R}, \#)$, that is, if $(A, <) \prec (\mathbb{R}, <)$, then $(A, \#) \prec (\mathbb{R}, \#)$. This follows from the fact that the relation $\#$ is definable in the structure $(\mathbb{R}, <)$.

For this reason, Section 3 furnishes many examples of elementary substructures of $(\mathbb{R}, \#)$.

We find more of them if we use Theorems 6.0.4 and 1.6.5.

6.1.1. Let A be a real subset of $\mathbb{R}$. We say that A is #-coherent of $\mathbb{R}$ if $\forall\alpha \in A\forall n \in \mathbb{N}\exists\beta \in A[\beta\#\alpha \wedge \beta(n) \approx \alpha(n)]$.

THEOREM 6.1.2. *Let A be an inhabited, real, stable and #-coherent subset of $\mathbb{R}$. Then $(A, \#) \prec (\mathbb{R}, \#)$.*

PROOF. Similar to the proof of Theorem 3.3.4. ⊣

6.1.3. It follows from Theorem 6.1.2 that the closed interval [0, 1] gives rise to an elementary substructure of $(\mathbb{R}, \#)$.

It is also true that the half-closed-half-open interval $[0, 1\rangle$ generates an elementary substructure of $(\mathbb{R}, \#)$. This is because the structure $([0, 1\rangle, \#)$ is isomorphic to the structure $([0, \infty\rangle, \#)$ and the half-line $[0, \infty\rangle$ generates an elementary substructure of $(\mathbb{R}, \#)$ by Theorem 6.1.2.

We now prove that the set $\langle -1, 0\rangle \cup [0, 1\rangle$ generates an elementary substructure of $(\mathbb{R}, \#)$.

Let $A := \langle -1, 0\rangle \cup [0, 1\rangle$. Let $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ be a finite sequence of elements of A.

We show:

$$\forall\beta \in \mathbb{R}\exists\alpha \in A[(\mathbb{R}, \#, \alpha_0, \alpha_1, \ldots, \alpha_{m-1}, \alpha) \equiv_0 (\mathbb{R}, \#, \alpha_0, \alpha_1, \ldots, \alpha_{m-1}, \beta)].$$

Without loss of generality, we may assume:

$$0 \leq k < m \text{ and } \forall i < k[\alpha_i \in \langle -1, 0\rangle \text{ and } \forall i \geq k[\alpha_i \in [0, 1\rangle].$$

Let $\beta \in \mathbb{R}$. As we observed in Section 3.1 we may decide: either $\beta \in \langle -1, 0\rangle$ or $\forall i < k[\alpha_i\#\beta]$. If $\beta \in \langle -1, 0\rangle$, we choose $\alpha := \beta$. If $\forall i < k[\alpha_i\#\beta]$, we determine γ in $[0, 1\rangle$ such that $(\mathbb{R}, \alpha_k, \ldots, \alpha_{m-1}, \beta) \equiv_0 (\mathbb{R}, \alpha_k, \ldots, \alpha_{m-1}, \gamma)$.

Observe that $\forall i < k[\gamma\#\alpha_i]$, and that we may choose $\alpha := \gamma$.

Using Theorems 6.0.4 and 1.6.5, we conclude that A generates an elementary substructure of $(\mathbb{R}, \#)$.

§7. Second intuitionistic intermezzo: the converse of Fraïssé's lemma fails.

7.0. We exhibit a pair of similar structures $\mathfrak{A} = (A, \ldots)$ and $\mathfrak{B} = (B, \ldots)$ such that $\mathfrak{A} \equiv_1 \mathfrak{B}$ and derive a contradiction from the assumption that $\forall a \in A\exists b \in B[(\mathfrak{A}, a) \equiv_0 (\mathfrak{B}, b)]$. Thus we see that the converse of Lemma 1.5, Fraïssé's Lemma, is constructively false.

Let $\mathfrak{A} := (\mathscr{N}, \{\mathbf{0}\})$ and $\mathfrak{B} := (\mathscr{N}, \{\alpha \in \mathscr{N} | \neg(\alpha = \mathbf{0})\})$. Observe that both stuctures satisfy the formula $\forall x[\neg\neg P(x) \to P(x)]$. Therefore, in $\mathfrak{A}$ as well as in $\mathfrak{B}$, each subset of $\mathscr{N}$ that may be defined by a quantifier-free formula, is defined by one of the following five formulas: $P(x)$, $\neg P(x)$, $P(x) \vee \neg P(x)$, $P(x) \wedge \neg P(x)$, $P(x) \to P(x)$.

It is now easily seen that $\mathfrak{A}$ satisfies the same sentences (that is, closed formulas) of quantifier-depth 1 as $\mathfrak{B}$, as both structures satisfy each one of the following ten formulas: $\exists x[P(x)]$, $\neg\forall x[P(x)]$, $\exists x[\neg P(x)]$, $\neg\forall x[\neg P(x)]$, $\exists x[P(x) \to P(x)]$, $\forall x[P(x) \to P(x)]$, $\neg\exists x[P(x) \wedge \neg P(x)]$, $\neg\forall x[P(x) \wedge \neg P(x)]$, $\exists x[P(x) \vee \neg P(x)]$ and $\neg\forall x[P(x) \vee \neg P(x)]$.

That both $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the last-mentioned formula follows from the continuity principle CP, see 5.0.0, as CP implies $\neg\forall\alpha \in \mathscr{N}[\alpha = \mathbf{0} \vee \neg(\alpha = \mathbf{0})]$, compare

Theorem 5.1. Observe that one does not need the continuity principle in order to see that $\mathfrak{A} \models \forall x[P(x) \vee \neg P(x)]$ iff $\mathfrak{B} \models \forall x[P(x) \vee \neg P(x)]$.

7.1. We introduce a stronger version of the continuity principle CP that we mentioned in Section 5.0.0. Observe that each function γ from $\mathbb{N}$ to $\mathbb{N}$, that is, each element γ of $\mathscr{N}$ may be viewed, via an enumeration of the set $\mathbb{N}^*$ of finite sequences of natural numbers, as a function from $\mathbb{N}^*$ to $\mathbb{N}^*$. We say that γ generates a (continuous) function from $\mathscr{N}$ to $\mathscr{N}$, notation $\mathrm{Fun}(\gamma)$, iff (i) for all $a, b \in \mathbb{N}^*$, if $a \sqsubseteq b$, then $\gamma(a) \sqsubseteq \gamma(b)$ and (ii) for each α in $\mathscr{N}$, m in $\mathbb{N}$ there exists n in $\mathbb{N}$ such that length $(\gamma(\overline{\alpha}(n)) > m$. In that case we define, for each α in $\mathscr{N}$, $\gamma{\scriptstyle `}\alpha$ to be the infinite sequence β of $\mathscr{N}$ such that for each n in $\mathbb{N}$, $\gamma(\overline{\alpha}(n))$ is an initial part of β. We are able now to formulate the stronger continuity principle:

7.1.0.

$\mathrm{AC}_{1,1}$ For every subset C of $\mathscr{N} \times \mathscr{N}$:
If $\forall \alpha \in \mathscr{N} \exists \beta \in \mathscr{N}[C(\alpha, \beta)]$,
then $\exists \gamma \in \mathscr{N}[\mathrm{Fun}(\gamma) \wedge \forall \alpha \in \mathscr{N}[C(\alpha, \gamma{\scriptstyle `}\alpha)]]$.

(Compare [4, *27.1] and $C - C$ in [6].)

7.2. We return to the structures $\mathfrak{A}, \mathfrak{B}$, introduced in Section 7.0. We have seen that $\mathfrak{A}$ satisfies the same first-order-sentences of quantifier-depth 1 as $\mathfrak{B}$, that is: $\mathfrak{A} \equiv_1 \mathfrak{B}$.

We claim: $\neg \forall \alpha \in \mathscr{N} \exists \beta \in \mathscr{N}[(\mathfrak{A}, \alpha) \equiv_0 (\mathfrak{B}, \beta)]$.

For suppose: $\forall \alpha \in \mathscr{N} \exists \beta \in \mathscr{N}[(\mathfrak{A}, \alpha) \equiv_0 (\mathfrak{B}, \beta)]$. Then, in particular, $\forall \alpha \in \mathscr{N} \exists \beta \in \mathscr{N}[\alpha = \mathbf{0} \rightleftarrows \neg(\beta = \mathbf{0})]$. Applying $\mathrm{AC}_{1,1}$ we find γ in $\mathbb{N}$ such that $\mathrm{Fun}(\gamma)$ and $\forall \alpha \in \mathscr{N}[\alpha = \mathbf{0} \rightleftarrows \neg(\gamma{\scriptstyle `}\alpha = \mathbf{0})]$. Consider $\gamma{\scriptstyle `}\mathbf{0}$ and assume m in $\mathbb{N}$ and $(\gamma{\scriptstyle `}\mathbf{0})(m) \neq 0$. Calculate $n$ in $\mathbb{N}$ such that $\forall \alpha \in \mathscr{N}[\overline{\alpha}(n) = \overline{\mathbf{0}}(n) \rightarrow (\gamma{\scriptstyle `}\alpha)(m) = (\gamma{\scriptstyle `}\mathbf{0})(m)]$. Then: $\forall \alpha \in \mathscr{N}[\overline{\alpha}(n) = \overline{\mathbf{0}}(n) \rightarrow \gamma{\scriptstyle `}\alpha \neq \mathbf{0}]$ and $\forall \alpha \in \mathscr{N}[\overline{\alpha}(n) = \overline{\mathbf{0}}n \rightarrow \alpha = \mathbf{0}]$. Contradiction. Therefore: $\gamma{\scriptstyle `}\mathbf{0} = \mathbf{0}$. Contradiction.

§8. Some structures elementarily equivalent to the structure $(\mathbb{R}, +)$.

We will see, among other things, that $(\mathbb{R}, +)$, the additive group of the real numbers, may be elementarily embedded into $(\mathbb{R}^2, +)$, the additive group of pairs of real numbers. We first formulate a useful model-theoretic lemma.

LEMMA 8.0 (Vaught's Lemma). *Let $\mathfrak{A} = (A, \ldots)$ be a mathematical structure and let $\mathfrak{B} = (B, \ldots)$ be a substructure of $\mathfrak{A}$. Suppose that for each n in $\mathbb{N}$, for each formula $\phi = \phi(x_0, x_1, \ldots, x_{n-1}, y)$ in the first-order-language of $\mathfrak{A}$, for each finite sequence $(b_0, b_1, \ldots, b_{n-1})$ of elements of B, for each a in A there exists b in B such that $\mathfrak{A} \models \phi[b_0, b_1, \ldots, b_{n-1}, a]$ iff $\mathfrak{A} \models \phi[b_0, b_1, \ldots, b_{n-1}, b]$. Then: $\mathfrak{B} \prec \mathfrak{A}$.*

PROOF. Like the proof of Lemma 1.5, Fraïssé's Lemma, the proof is straightforward and constructive. ⊣

8.1. We study the structure $(\mathbb{R}, +)$. We will not consider $(\mathbb{R}, +)$ itself but prefer its "relational variant" $(\mathbb{R}, \mathrm{Add})$ where Add denotes the set $\{(\alpha, \beta, \gamma) \in \mathbb{R}^3 \mid \alpha + \beta \# \gamma\}$. We do so in order to have simple basic formulas. One might also restrict attention to so-called *unnested formulas* in the first-order-language of $(\mathbb{R}, +)$, see [3].

For each n in $\mathbb{N}$, and each formula $\phi = \phi(x_0, x_1, \ldots, x_{n-1})$ in the first-order-language of $(\mathbb{R}, \mathrm{Add})$ we define an equivalence relation $\sim_\phi$ on the set $\mathbb{R}^n$ by:

$(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \sim_\phi (\beta_0, \beta_1, \ldots, \beta_{n-1})$ iff: $((\mathbb{R}, \mathrm{Add}) \models \phi[\alpha_0, \alpha_1, \ldots, \alpha_{n-1}]$ iff $(\mathbb{R}, \mathrm{Add}) \models \phi[\beta_0, \beta_1, \ldots, \beta_{n-1}])$.

We also introduce, for each n in $\mathbb{N}$, and each finite set A of rational numbers an equivalence relation $\sim_A$ on the set $\mathbb{R}$ by: $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \sim_A (\beta_0, \beta_1, \ldots, \beta_{n-1})$ iff for all $q_0, q_1, \ldots, q_{n-1}$ in A : $\sum_{j<n} q_j \cdot \alpha_j \# 0$ iff $\sum_{j<n} q_j \cdot \beta_j \# 0$. We claim that for each formula $\phi = \phi(x_0, x_1, \ldots, x_{n-1})$ from the first-order-language of $(\mathbb{R}, \mathrm{Add})$ there exists a finite set A of rational numbers such that $\sim_A \subseteq \sim_\phi$, that is, such that for all finite sequences $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ and $(\beta_0, \beta_1, \ldots, \beta_{n-1})$ of elements of $\mathbb{R}$: If $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \sim_A (\beta_0, \beta_1, \ldots, \beta_{n-1})$, then $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \sim_\phi (\beta_0, \beta_1, \ldots, \beta_{n-1})$.

We prove this claim by induction on $QD(\phi)$.

If ϕ is a quantifier-free formula, that is, if $QD(\phi) = 0$, we may take $A := \{0, 1, -1, 2, -2\}$.

Now suppose that ϕ is an existential formula, say

$$\exists y[\psi(x_0, x_1, \ldots, x_{n-1}, y)],$$

or a universal formula, say

$$\forall y[\psi(x_0, x_1, \ldots, x_{n-1}, y)],$$

and that A is a finite set of rational numbers such that $\sim_A \subseteq \sim_\psi$. We assume that 0 and 1 belong to A and that for each q in $\mathbb{Q}$, if q belongs to A, then also $-q$ belongs to A. We now define $B := \{\frac{q}{s} - \frac{r}{t} \mid q, r, s, t \in A, s \neq 0, t \neq 0\}$ and we show that $\sim_B \subseteq \sim_\phi$. Let $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ and $(\beta_0, \beta_1, \ldots, \beta_{n-1})$ be finite sequences of elements of $\mathbb{R}$ such that $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \sim_B (\beta_0, \beta_1, \ldots, \beta_{n-1})$. Observe that for all $q_0, q_1, \ldots, q_{n-1}, r_0, r_1, \ldots, r_{n-1}, s, t$ in A such that $s \neq 0$ and $t \neq 0$ the following holds: $\sum_{j<n} \frac{q_j}{s} \cdot \alpha_j \# \sum_{j<n} \frac{r_j}{t} \cdot \alpha_j$ iff $\sum_{j<n} \frac{q_j}{s} \cdot \beta_j \# \sum_{j<n} \frac{r_j}{t} \beta_j$.

We now use the fact that the structure $(\mathbb{R}, \#)$ is back-and-forth-homogeneous, see Theorem 6.0.3. We may construct, for every α in $\mathbb{R}$ an element β of $\mathbb{R}$ such that for all $q_0, q_1, \ldots, q_{n-1}, s$ in A such that $s \neq 0$ the following holds:

$$\sum_{j<n} \frac{q_j}{s} \cdot \alpha_j \# \alpha \text{ iff } \sum_{j<n} \frac{q_j}{s} \cdot \beta_j \# \beta,$$

that is: $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \alpha) \sim_A (\beta_0, \beta_1, \ldots, \beta)$ and therefore: $(\mathbb{R}, \mathrm{Add}) \models \psi[\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \alpha]$ if and only if $(\mathbb{R}, \mathrm{Add}) \models \psi[\beta_0, \beta_1, \ldots, \beta_{n-1}, \beta]$.
Applying Lemma 1.2 we conclude:

If $(\mathbb{R}, \mathrm{Add}) \models \exists x[\psi][\alpha_0, \alpha_1, \ldots, \alpha_{n-1}]$, then $(\mathbb{R}, \mathrm{Add}) \models \exists x[\psi][\beta_0, \beta_1, \ldots, \beta_{n-1}]$

and:

If $(\mathbb{R}, \mathrm{Add}) \models \forall x[\psi][\beta_0, \beta_1, \ldots, \beta_{n-1}]$, then $(\mathbb{R}, \mathrm{Add}) \models \forall x[\psi][\alpha_0, \alpha_1, \ldots, \alpha_{n-1}]$.

One proves in the same way:

If $(\mathbb{R}, \mathrm{Add}) \models \exists x[\psi][\beta_0, \beta_1, \ldots, \beta_{n-1}]$, then $(\mathbb{R}, \mathrm{Add}) \models \exists x[\psi][\alpha_0, \alpha_1, \ldots, \alpha_{n-1}]$

and:

If $(\mathbb{R}, \mathrm{Add}) \models \forall x[\psi][\alpha_0, \alpha_1, \ldots, \alpha_{n-1}]$, then $(\mathbb{R}, \mathrm{Add}) \models \forall x[\psi][\beta_0, \beta_1, \ldots, \beta_{n-1}]$.

8.2. We mention some structures that are elementarily equivalent to $(\mathbb{R}, \mathrm{Add})$.

8.2.1. Consider $D := \{\alpha \in \mathbb{R} \mid \neg\neg\exists q \in \mathbb{Q}[\alpha = i(q)]\}$, that is, the set of not-not-rational numbers. We claim that $(D, \mathrm{Add}) \prec (\mathbb{R}, \mathrm{Add})$. We prove this claim by applying Lemma 8.0. Let $\phi = \phi(x_0, x_1, \ldots, x_{n-1}, y)$ be a formula in the first-order-language of $(\mathbb{R}, \mathrm{Add})$. Using the results of Section 8.2, determine a finite set A of rational numbers containing 0,1, such that $\sim_A \subseteq \sim_\phi$. Now, let $\beta_0, \beta_1, \ldots, \beta_{n-1}$ be a finite sequence of elements of D, and let $\alpha \in \mathbb{R}$. From the proof of Theorem 3.3.4 we see that it is possible to find β in D such that for all $q_0, q_1, \ldots, q_{n-1}, s$ in A, if $s \neq 0$ then: $\sum_{j<n} \frac{q_j}{s}\beta_j \# \alpha$ if and only if $\sum_{j<n} \frac{q_j}{s}\beta_j \# \beta$, and therefore, $(\beta_0, \beta_1, \ldots, \beta_{n-1}, \alpha) \sim_A (\beta_0, \beta_1, \ldots, \beta_{n-1}, \beta)$, and therefore: $(\mathbb{R}, \mathrm{Add}) \models \phi[\beta_0, \beta_1, \ldots, \beta_{n-1}, \alpha]$ if and only if $(\mathbb{R}, \mathrm{Add}) \models \phi(\beta_0, \beta_1, \ldots, \beta_{n-1}, \beta]$. This concludes the proof of our claim. Observe that also $(D, \mathrm{Add}, <) \prec (\mathbb{R}, \mathrm{Add}, <)$.

8.2.2. Generalizing the example given in Section 8.2.1 we may take any subset A of $\mathbb{R}$ that gives rise to a divisible subgroup of $(\mathbb{R}, \mathrm{Add})$ and contains an element apart from 0. As such a set A is dense in $\mathbb{R}$ we may conclude, as in Section 8.2.1, that $\{\alpha \in \mathbb{R} \mid \neg\neg\exists \beta \in A[\beta \equiv \alpha]\}$ gives rise to an elementary substructure of $(\mathbb{R}, \mathrm{Add})$.

8.2.3. Mutatis mutandis, what we said on the structure $(\mathbb{R}, +)$ may be said on the structure $(\mathbb{R}^2, +)$. Consider $A = \{(\alpha, \alpha) \mid \alpha \in \mathbb{R}\}$ and observe that A is a stable subset of $\mathbb{R}^2$ as $\forall\alpha \in \mathbb{R}\forall\beta \in \mathbb{R}[\neg\neg(\alpha \equiv \beta) \to \alpha \equiv \beta]$. Moreover, A is a #-coherent subset of $\mathbb{R}^2$, (see Definition 6.3.1) if we adapt this notion to the structure $(\mathbb{R}^2, \#)$. Therefore, $(A, +)$ is an elementary substructure of $(\mathbb{R}^2, +)$ and $\alpha \mapsto (\alpha, \alpha)$ is an elementary embedding of the structure $(\mathbb{R}, +)$ into the structure $(\mathbb{R}^2, +)$.

8.2.4. Consider the structure $(\mathbb{Q}^{\mathbb{N}}, +)$ where addition of infinite sequences of rational numbers is defined component-wise. $(\mathbb{Q}^{\mathbb{N}}, +)$ is elementarily equivalent to $(\mathbb{R}, +)$. One may prove this by building the product structure $(\mathbb{Q}^{\mathbb{N}} \times \mathbb{R}, +)$ and then observing that both $\mathbb{Q}^{\mathbb{N}} \times \{0\}$ and $\{\mathbf{0}\} \times \mathbb{R}$ (where $\mathbf{0}$ denotes the neutral element of $(\mathbb{Q}^{\mathbb{N}}, +)$) are stable and #-coherent subsets of $\mathbb{Q}^{\mathbb{N}} \times \mathbb{R}$ and thus give rise to elementary substructures of $(\mathbb{Q}^{\mathbb{N}} \times \mathbb{R}, +)$. As in 8.2.3 we leave the details of the proof to the reader.

8.3. We make a minor final remark. One might be tempted to consider, for each n in $\mathbb{N}$, the equivalence relation $\sim^n_{\mathbb{Q}}$ on the set $\mathbb{R}^n$, that is given by: $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \sim^n_{\mathbb{Q}} (\beta_0, \beta_1, \ldots, \beta_{n-1})$ iff for all rational numbers $q_0, q_1, \ldots, q_{n-1}$: $\sum_{j<n} q_j \cdot \alpha_j \# 0$ iff $\sum_{j<n} q_j \cdot \beta_j \# 0$. It follows from our remarks in Section 8.2 that (intuitionistically no less than classically) $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \sim^n_{\mathbb{Q}} (\beta_0, \beta_1, \ldots, \beta_{n-1})$ iff for each formula $\phi = \phi(x_0, x_1, \ldots, x_{n-1})$ in the first-order-language of $(\mathbb{R}, \mathrm{Add})$: $(\mathbb{R}, \mathrm{Add}) \models \phi[\alpha_0, \alpha_1, \ldots, \alpha_{n-1}]$ iff $(\mathbb{R}, \mathrm{Add}) \models \phi[\beta_0, \beta_1, \ldots, \beta_{n-1}]$.

We show that, constructively, the sequence of equivalence-relations $\sim^0_{\mathbb{Q}}, \sim^1_{\mathbb{Q}}, \sim^2_{\mathbb{Q}}, \ldots$ does *not* enjoy the back-and-forth-property.

We construct a Brouwerian counter-example. (In fact, this is our intuitionistic postludium.) Let $d : \mathbb{N} \to \{0, 1, \ldots, 9\}$ be the function such that $\pi = 3 + \sum_{n=0}^{\infty} d(n) \cdot 10^{-n-1}$, that is, d is the decimal expansion of π. We construct real numbers α and β as follows. For each n in $\mathbb{N}$, if $\neg\exists m < n\forall i < 99[d(m+i) = 9]$, $\alpha(n) := \beta(n) := (-\frac{1}{n}, \frac{1}{n})$, and if $\exists m < n\forall i < 99[d(m+i) = 9]$ and $k = \mu m < n\forall i < 99[d(m+i) = 9]$, then, if k is odd $\alpha(n) := (\frac{1}{k}, \frac{1}{k})$ and $\beta(n) := (\frac{1}{2k}, \frac{1}{2k})$ and if k is even $\alpha(n) := (\frac{1}{2k}, \frac{1}{2k})$ and $\beta(n) := (\frac{1}{k}, \frac{1}{k})$. Observe that for each rational

number $q : q \cdot \alpha \# 0$ if and only if $q \cdot \beta \# 0$, that is: $(\alpha) \sim^1_{\mathbb{Q}} (\beta)$. Now assume: $\gamma \in \mathbb{R}$ and $(\alpha, 1) \sim^2_{\mathbb{Q}} (\beta, \gamma)$. Observe that, if $\exists m \forall i < 99[d(m+i) = 9]$ and the least such m is odd, then $\alpha \equiv 2 \cdot \beta$ and $\gamma \equiv \frac{1}{2}$, but, if $\exists m \forall i < 99[d(m+i) = 9]$ and the least such m is even, then $\beta \equiv 2 \cdot \alpha$ and $\gamma \equiv 2$. This shows that we are unable to find γ in $\mathbb{R}$ such that $(\alpha, 1) \sim^2_{\mathbb{Q}} (\beta, \gamma)$.

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MATHEMATISCH INSTITUUT
KATHOLIEKE UNIVERSITEIT
6525 ED NIJMEGEN
THE NETHERLANDS

E-mail: veldman@sci.kun.nl
waaldijk@sci.kun.nl